Chapter 7

Rotational Dynamics

2. Kinetic Energy of Rigid Body

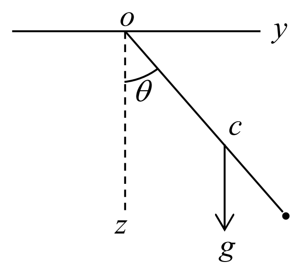
The rigid body of mass M and radius R. The moment of inertia about center of mass is I_{cm}. The velocity of center of mass is v_{cm} and angular velocity about center of mass is ω, then kinetic energy is $T = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$

Example: A sphere of mass M and radius R having velocity of center of mass v_0. As friction of surface is enough so that sphere Achieving condition of Rolling without Slipping. Find the kinetic energy of sphere.

Solution: $T = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}Mv_0^2 + \frac{1}{2}\frac{2}{5}MR^2\omega^2$

Rolling without slipping, $v_{c.m} = \omega R \quad \omega = \frac{v_0}{R} \quad \frac{1}{2}Mv_0^2 + \frac{1}{2}\frac{2}{5}MR^2\frac{v_0^2}{R^2} = \frac{7}{10}Mv_0^2$

Example: A rod of length l is pivoted at one of its end. If rod is constrained to move in $y-z$ plane, then



(a) find the total energy of system

(b) find angular frequency for small oscillation.

Solution: (a) $T=\frac{1}{2}M\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}I_{cm}\omega^2$

$$x=\frac{l}{2}\sin\theta,\Rightarrow \dot{x}=\frac{l}{2}\cos\theta\dot{\theta}\quad y=\frac{l}{2}\cos\theta\Rightarrow \dot{y}=-\frac{l}{2}\sin\theta\dot{\theta}$$

Kinetic energy $T=\frac{1}{2}\frac{l^2}{4}\dot{\theta}^2+\frac{1}{2}\frac{Ml^2}{12}\dot{\theta}^2=\frac{Ml^2}{6}\dot{\theta}^2$

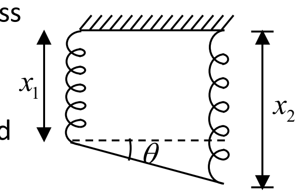
Potential energy V is given by $V(\theta)=-\frac{Mgl\cos\theta}{2}$

$$E=T+V\quad\Rightarrow E=\frac{1}{2}\left(\frac{Ml^2}{3}\right)\dot{\theta}^2-\frac{Mgl}{2}\cos\theta$$

(b) Hence, energy is conserved $\frac{dE}{dt}=0\Rightarrow E=\left(\frac{Ml^2}{3}\right)\dot{\theta}\ddot{\theta}+\frac{Mgl}{2}\sin\theta\dot{\theta}=0\Rightarrow\left(\frac{l}{3}\right)\ddot{\theta}+\frac{g}{2}\sin\theta=0$

for small oscillation, $\sin\theta=\theta$, so $\left(\frac{l}{3}\right)\ddot{\theta}+\frac{g}{2}\theta=0\Rightarrow\omega=\sqrt{\frac{3g}{2l}}$

Example: A rod of mass m and length l is suspended from two mass less vertical springs with a spring constants k_1 and k_2. If x_1 and x_2 be the displacements from equilibrium position of the two ends of the rod, find the kinetic energy of system.



Solution: $T=\frac{1}{2}Mv_{c.m}^2+\frac{1}{2}I_{c.m}\omega^2=\frac{1}{2}m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2+\frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2$

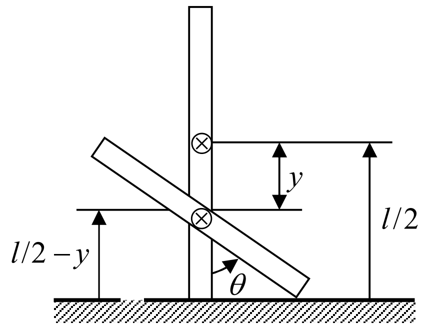
$\sin\theta=\frac{x_2-x_1}{l}$ for small oscillation $\theta=\frac{x_2-x_1}{l}\Rightarrow\dot{\theta}=\frac{\dot{x}_2-\dot{x}_1}{l}$

(keep in mind that there is not pure rotation case)

$$T=\frac{1}{2}m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2+\frac{1}{2}\frac{ml^2}{12}\left(\frac{\dot{x}_1-\dot{x}_2}{l}\right)^2\Rightarrow T=\frac{m}{6}\left(\dot{x}_1^2+\dot{x}_1\dot{x}_2+\dot{x}_2^2\right)$$

Example: A stick of length l and mass M initially upright on a frictionless table, starts falling. The problem is to find the speed of the center of mass as a function of position.

Solution: The key lies in realizing that since there are no horizontal forces, the center of mass must fall straight down. Since we must find velocity as a function of position, it is natural to apply energy methods.



The sketch shows the stick after it has rotated through angle θ and the center of mass has fallen distance y. The total energy at rest is $E = K_0 + U_0 = \dfrac{Mgl}{2}$

The kinetic energy at a later time is $K = \dfrac{1}{2} I_{cm}\dot{\theta}^2 + \dfrac{1}{2} M\dot{y}^2$

and the corresponding potential energy is $U = Mg\left(\dfrac{l}{2} - y\right)$.

Since, there are no dissipative forces, mechanical energy is conserved and

$$K + U = K_0 + U_0 = \frac{Mgl}{2}.$$

Hence, $\dfrac{1}{2}M\dot{y}^2 + \dfrac{1}{2}I_{cm}\dot{\theta}^2 + Mg\left(\dfrac{l}{2} - y\right) = Mg\dfrac{l}{2}$.

We can eliminate θ by turning to the constraint equation. From the sketch we see that $y = \dfrac{l}{2}(1 - \cos\theta)$. Hence, $\dot{y} = \dfrac{l}{2}\sin\theta\dot{\theta}$ and $\dot{\theta} = \dfrac{2}{l\sin\theta}\dot{y}$ Since, $I_{cm} = M\left(\dfrac{l^2}{12}\right)$, we obtain

$$\frac{1}{2}M\dot{y}^2 + \frac{1}{2}M\frac{l^2}{12}\left(\frac{2}{l\sin\theta}\right)^2\dot{y}^2 + Mg\left(\frac{l}{2} - y\right) = Mg\frac{l}{2}$$

or $$\dot{y}^2 = \frac{2gy}{\left[1 + \dfrac{1}{\left(3\sin^2\theta\right)}\right]} \Rightarrow \dot{y} = \left[\frac{6gy\sin^2\theta}{3\sin^2\theta + 1}\right]^{1/2}, \quad \text{where } y = \frac{l}{2}(1 - \cos\theta)$$