Chapter 6

Linear Momentum And Energy

3. Energy

In Physics, **Energy** is the quantitative property that must be transferred to an object in order to perform work on, or to transfer heat to the object Energy is a conserved quantity; the law of conservation of energy states that energy can be converted in form, but not created or destroyed. The SI unit of energy is the joule, which is the energy transferred to an object by the work of moving it a distance of 1 metre against a force of 1 Newton.

Conservation of Energy: The energy can neither will created nor destroy, it can be transform from one form to another.

Different Form of Energy used in Mechanics

Kinetic energy

The energy which is responsible for motion of the particle. If particle of mass m moving with velocity $\vec{v}$, then kinetic energy is given by $K=\frac{1}{2}\vec{p}.\vec{v}=\frac{1}{2}m\left(\vec{v}.\vec{v}\right)$.

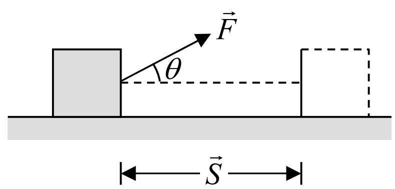
Kinetic energy in Cartesian coordinate $K=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$

Kinetic energy in cylindrical coordinate $K=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2)$

Kinetic energy in spherical coordinate $K=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2)$

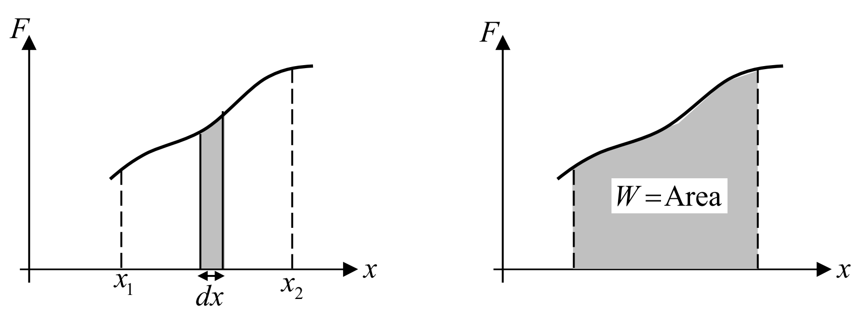
Work Done by Force

Work done by the constant force is given as $W=\vec{F}\cdot\vec{S}=FS\cos\theta$, where force $\vec{F}$ is force and making θ angle with displacement vector $\vec{S}$. In general work done is a area under the force F and displacement S, which is for the variable force is defined as $W=\int\vec{F}\cdot\vec{d}S$



Work Done by a Variable Force

So far we have considered the work done by a force which is constant both in magnitude and direction. Let us now consider a force which acts always in one direction but whose magnitude may keep on varying. We can choose the direction of the force as x - axis. Further, let us assume that the magnitude of the force is also a function of x or say $F\left(x\right)$ is known to us. Now we are interested in finding the work done by this force in moving a body from x_1 to x_2.



Work done in a small displacement from x and $x+dx$ will be

$$dW=F\cdot dx$$

Now, the total work can be obtained by integration of the above elemental work from x_1 to x_2

or $W=\int_{x_1}^{x_2} dW=\int_{x_1}^{x_2} F\cdot dx$ It is important to note that $\int_{x_1}^{x_2} F\cdot dx$ is also the area under F-x graph between $x=x_1$ to $x=x_2$.

Potential energy (U)

The energy which is required to perform the work is known as potential energy U. Hence force is defined as $F=-\frac{\partial U}{\partial r}$. Then potential energy $U_b-U_a=-\int_a^b \vec{F}.d\vec{r}$. For the conservative force one can say potential energy is negative integral of the force .one can say in another way change in potential energy with respect to position is cause of force F.

There is different type of potential energy. For example, electrostatic potential energy, gravitation potential energy, stored energy in spring, mass energy in relativistic mechanics.

Total energy E is sum of kinetic energy K and potential energy U. So total energy $E=K+U$

Conservative and Non Conservative Force Field

We considered the forces which were although variable but always directed in one dimension. However, the most general expression for work done is

$$dW=\vec{F}\cdot\vec{dr} \text{ and } W=\int_{\vec{r}_i}^{\vec{r}_f} dW=\int_{\vec{r}_i}^{\vec{r}_f}\vec{F}\cdot\vec{dr}\text{. Here, } \vec{dr}=dx\hat{i}+dy\hat{j}+dz\hat{k}$$

$\vec{r}_i=$ initial position vector and $\vec{r}_f=$ final position vector

A **conservative force** is a force with the property that the total work done in moving a particle between two points is independent of the taken path. Equivalently, if a particle travels in a closed loop, the total work done (the sum of the force acting along the path multiplied by the displacement) by a conservative force is zero.

For example Gravitational force is an example of a conservative force, while frictional force is an example of a non-conservative force.

Mathematical Interpretation of Conservative Force

A Force field $\vec{F}$, defined everywhere in space (or within a simply connected volume of space), is called a *conservative force* or *conservative* if it meets any of these three *equivalent* conditions:

(1) The curl of $\vec{F}$ is the zero vector: $\vec{\nabla}\times\vec{F}=0$

(2) There is zero net work (*W*) done by the force when moving a particle through a trajectory that starts and ends in the same place $\oint \vec{F}.d\vec{r}=0$

(3) The force can be written as the negative gradient of a potential , $\vec{F}=-\vec{\nabla}V$:

Example: A body is displaced from $A=(2m,4m,-6m)$ to $\vec{r}_B=\left(6\hat{i}-4\hat{j}+2\hat{k}\right)m$ under a constant force $\vec{F}=\left(2\hat{i}+3\hat{j}-\hat{k}\right)N$. Find the work done.

Solution: $\vec{r}_A=\left(2\hat{i}+3\hat{j}-\hat{k}\right)m$

$$\vec{S}=\vec{r}_B-\vec{r}_A=\left(6\hat{i}-4\hat{j}+2\hat{k}\right)-\left(2\hat{i}+4\hat{j}-6\hat{k}\right)=4\hat{i}-8\hat{j}+8\hat{k}$$

$$W=\vec{F}\cdot\vec{S}=\left(2\hat{i}+3\hat{j}-\hat{k}\right)\cdot\left(4\hat{i}-8\hat{j}+8\hat{k}\right)=8-24-8=-24\,J$$

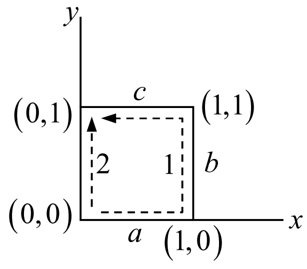
Example: A force $F=-\frac{k}{x^2}(x\neq 0)$ acts on a particle in x- direction. Find the work done by this force in displacing the particle from $x=+a$ to $x=+2a$. Here, k is a positive constant.

Solution: $W=\int F dx=\int_{+a}^{+2a}\left(\frac{-k}{x^2}\right)dx=\left[\frac{k}{x}\right]_{+a}^{+2a}=-\frac{k}{2a}$

Work done by area under F-S or F-x Graph

Example: (a) Prove that for $F=A\left(xy\hat{i}+y^2\hat{j}\right)$ is non conservative.

(b) Consider the integral $W=\int\vec{F}.dr$ from $(0,0)$ to $(0,1)$, first along path 1 and then along path 2, as shown in the figure. Check whether it is path independent or not.



Solution: (a) $\vec{\nabla}\times\vec{F}=-Ax\hat{k}\neq 0$. So force is non conservative.

(b) The force F has no physical significance, but the example illustrates the properties of non conservative forces. Since the segments of each path lie along a coordinate axis, it is particularly simple to evaluate the integrals. For path 1 we have

$$\oint_1 F \cdot dr = \int_a F \cdot dr + \int_b F \cdot dr + \int_c F \cdot dr$$

Along segment $a dr = dx\hat{i}, F \cdot dr = F_x dx = Axydx$. Since $y = 0$ along the line of this integration, $\int_a F \cdot dr = 0$. Similarly, for path b, $\int_b F \cdot dr = A \int_{x=1,\, y=0}^{x=1,\, y=1} y^2 dy = \frac{A}{3}$,

while for path c, $\int_c F \cdot dr = A \int_{x=1,\, y=1}^{x=0,\, y=1} xy\, dx = A \int_1^0 x\, dx = -\frac{A}{2}$

Thus, $\oint_1 F \cdot dr = \frac{A}{3} - \frac{A}{2} = -\frac{A}{6}$

Along path 2 we have $\oint_2 F \cdot dr = A \int_{0,0}^{0,1} y^2 dy = \frac{A}{3} \neq \oint_1 F \cdot dr$

Example: The field is given as $\vec{F} = k\left(x\hat{x} + y\hat{y} + z\hat{z}\right)$, where k is positive constant, then check whether the field is conservative or non conservative.

Solution: For conservative field $\vec{\nabla} \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ x & y & z \end{vmatrix} = \hat{x}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{y}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{z}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

Which justify the following force is conservative in nature.