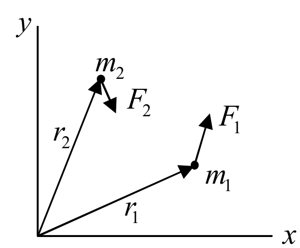
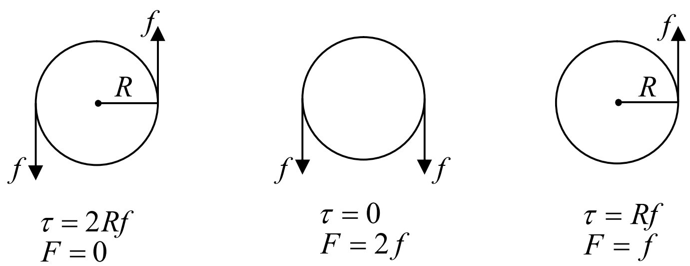
Chapter 7

Rotational Dynamics

3. Torque

The torque $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{\tau}$ and $\vec{F}$ are always perpendicular. There can be a torque on a system with zero net force, and there can be force with zero net torque. In general, there will be both torque and force. These three cases are illustrated in the sketches as shown below. (The torques are evaluated about the centers of the disks.)





If $\vec{\tau}_0$ is the component of the torque about the center of mass and $\vec{F}$ is the total applied force, then torque is given by $\vec{\tau} = \vec{\tau}_0 + \left(\vec{r} \times \vec{F}\right)$

The first term is the torque about the center of mass due to the various external forces, and the second term is the torque due to the total external force acting at the center of mass.

Conservation of Angular Momentum

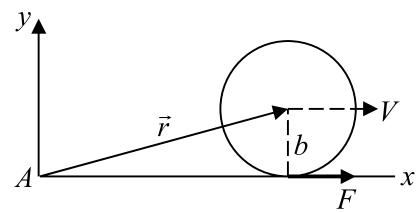
Torque is important because it is intimately related to the rate of change of angular momentum:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

But $\left(d\vec{r}/dt\right) \times \vec{p} = \vec{v} \times m\vec{v} = 0$, since the cross product of two parallel vectors is zero. Also, $d\vec{p}/dt = \vec{F}$, by Newton's second law. Hence, the second terms is $\vec{r} \times \vec{F} = \vec{\tau}$, and we have $\vec{\tau} = \frac{d\vec{L}}{dt}$,

If external torque is zero $\vec{\tau} = \frac{d\vec{L}}{dt} = 0$ then angular momentum, $\vec{L}$ is constant and the angular momentum is conserved.

Example: A disk of mass M and radius b is pulled with constant force F by a thin tape wound around its circumference.



The disk slides on ice without friction.

(a) What will be torque about Center of mass

(b) What will be torque about Point A

(c) If I_{cm} is moment of inertia about center of mass. Find the acceleration of center of mass from result obtain by a)

(d) Find the acceleration of center of mass from result obtain by (a)

Solution: (a) $\vec{\tau} = \vec{\tau}_0 + \left(\vec{r} \times \vec{F}\right)$ where $\tau_0 = bF\,\hat{z}$ and $\left(\vec{r} \times \vec{F}\right) = 0$ because $\vec{r} = 0$ as center of mass is origin

(b) Choose a coordinate system whose origin A is along the line of F. The torque about center of mass C is, $\tau_0 = bF\hat{z}$

$$\left(\vec{r} \times \vec{F}\right) = |r| F \sin\theta\left(-\hat{z}\right) = -bF\hat{z}, \quad \vec{\tau}_z = \vec{\tau}_0 + \left(\vec{r} \times \vec{F}\right)_z = bF - bF = 0$$

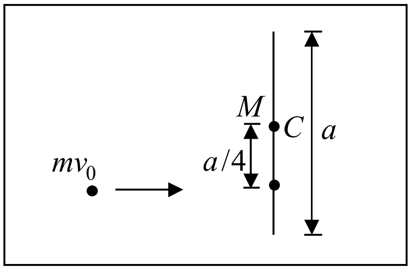
(c) $\tau_{cm} = I_{cm}\alpha \Rightarrow bF = I_{cm}\alpha \Rightarrow \alpha = \dfrac{bF}{I_{cm}}$

(d) The torque about A is zero. As we expect that angular momentum about the origin is conserved.

The angular momentum about A is $L_z = I_{cm}\omega + (R \times MV)_z = I_{cm}\omega - bMV$.

Since, $\dfrac{dL_z}{dt} = 0$, we have $0 = I_{cm}\alpha - bMa$ or $\alpha = \dfrac{bMa}{I_{cm}} = \dfrac{bF}{I_{cm}}$, as before.

Example: A uniform rod of mass and length a lies on smooth horizontal plane. A particle of mass m moving at speed v_0 perpendicular to length of rod strikes it at a distance $\dfrac{a}{4}$ from the center and stops after the collision.

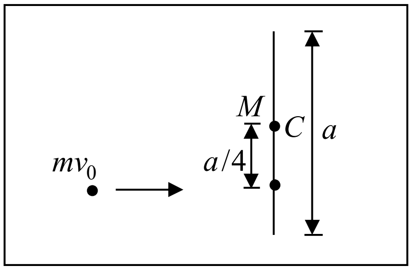


(a) Find the velocity of center of mass of rod

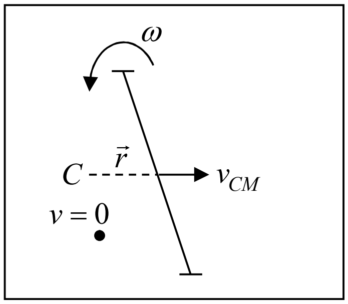
(b) Find the angular velocity of rod about its center of mass just after collision.

Solution: From conservation of linear momentum

$$mv_0 + M \times 0 = m \times 0 + Mv_{cm}$$



System before collision



System after collision

In this problem Angular momentum is conserve about any point because external torque is zero so from conservation of angular momentum about center of mass

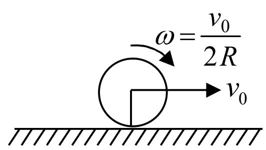
$mv_0 \times \dfrac{a}{4} + 0 = 0 + M(\vec{r} \times \vec{v}_{cm}) + I_{cm}\omega$ about origin $(\vec{r} \times \vec{v}_{cm}) = 0$ $\vec{r}$ is parallel to $\vec{v}_{cm}$

$$mv_0 \times \frac{a}{4} + 0 = 0 + 0 + I_{cm}\omega \Rightarrow \frac{mv_0 a}{4} = \frac{Ml^2}{12}\omega \Rightarrow \omega = \frac{3mv_0}{Ma}$$

Example: A sphere of mass M and radius R having velocity of center of mass v_0 and angular velocity $\frac{v_0}{2R}$. The friction of surface are enough that after some time sphere achieve the case of rolling without slipping. Find the velocity of center of mass.

Solution: In this problem, the angular momentum is conserved about point of contact P.

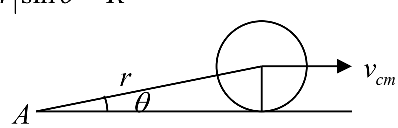
Initial angular momentum is



$$M\left(\vec{r}\times\vec{v}_{cm}\right)+I_{cm}\vec{\omega}=Mv_0R+\frac{2}{5}MR^2\frac{v_0}{2R}=\frac{6}{5}MRv_0$$

The angular momentum about point P when sphere achieve the condition of rolling without slipping

$$M\left(\vec{r}\times\vec{v}_{cm}\right)+I_{cm}\vec{\omega}=M\left|r\right|v_{cm}\sin\theta+\frac{2}{5}MR^2\omega\ ,\quad \left|r\right|\sin\theta=R$$



$$Mv_{cm}R+\frac{2}{5}MR^2\frac{v_{cm}}{R}=\frac{7}{5}MRv_{cm}$$

From conservation of angular momentum

$$\frac{6}{5}MRv_0=\frac{7}{5}MRv_{cm}\Rightarrow v_{cm}=\frac{6}{7}v_0$$