Chapter 7

Rotational Dynamics

4. Newton's Law of Motion for Rigid Body

A rigid body having mass M and moment of inertia about center of mass is I_{cm}

External Force $\vec{F}$ is acting on center of mass and $\vec{a}_{cm}$ is linear acceleration of center of mass.

From Newton's law of motion, $\vec{F} = M\vec{a}_{cm}$

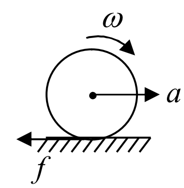
Torque about center of mass is $\vec{\tau} = I_{cm}\vec{\alpha}$ where $\vec{\alpha}$ is angular acceleration about center of mass

For rolling without slipping, $\alpha = \frac{a_{cm}}{R}$

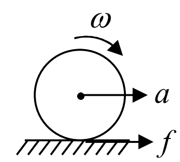
Direction of frictional force in case of rotational dynamics

Assume a sphere of mass M and radius R are rolling on a rough surface, where a is acceleration of center of mass and α is angular acceleration.

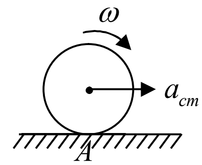
Case 1: If acceleration of center of mass a is more than $R\alpha$, then friction will opposite to direction of acceleration of center of mass F



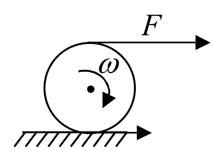
Case 2: If acceleration of center of mass a is less than $R\alpha$, then friction will same to direction of acceleration of center of mass



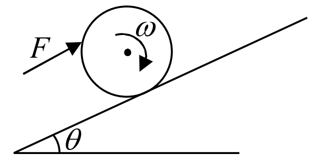
Case 3: In case of rolling without slipping point of contact A will remain at rest, then the direction of frictional force depends on applied torque and possible relative velocity between surface and sphere.



For example if applied force F is at topmost point of sphere then due to torque lowest point will move clock wise direction then frictional force is in same direction of F.



In another example, if sphere is rolling up on inclined plane and by force F and sphere is rolling upward so point of contact A is tend to move clockwise direction to keep this point rest frictional force will in upward direction.

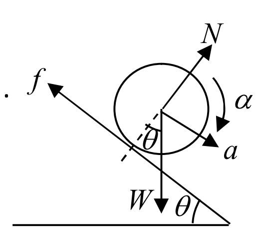


Example: A uniform drum of radius R and mass M rolls without slipping down a plane inclined at angle θ. Find its acceleration along the plane. The moment of inertia of the drum about its axis is $I_0 = \frac{MR^2}{2}$.

Solution: The forces acting on the drum are shown in the diagram. f is the force of friction. The translation of the center of mass along the plane is given by

$$W \sin\theta - f = Ma$$

and the rotation about the center of mass by torque equation $Rf = I_0\alpha$.



For rolling without slipping, we also have

$$a = R\alpha.$$

If we eliminate f, we obtain $W\sin\theta - I_0\dfrac{\alpha}{b} = Ma$

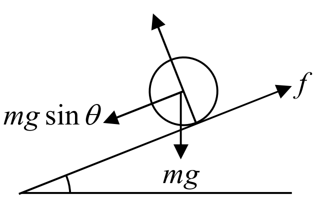
Using $I_0 = \dfrac{MR^2}{2}$, and $\alpha = \dfrac{a}{R}$, we obtain

$Mg\sin\theta - \dfrac{Ma}{2} = Ma$ or $a = \dfrac{2}{3}g\sin\theta$.

Example: A sphere of mass M and radius R rolling down the wedge with angle θ. Find the minimum value of coefficient of static friction to support pure rolling.

Solution: Force equation $Mg\sin\theta - f = Ma$ ....(i)

The torque equation is given by $\tau = I_{cm}\alpha$



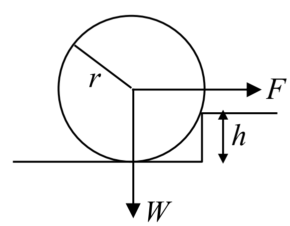
$fR = I_{cm}\alpha$ where $\alpha = \dfrac{a}{R}$ $I_{cm} = \dfrac{2}{5}MR^2$

$$f = \frac{2}{5}Ma \Rightarrow Ma = \frac{5}{2}f \qquad \text{....(ii)}$$

$Mg\sin\theta - f = \dfrac{5}{2}f \Rightarrow f = \dfrac{2}{7}Mg\sin\theta$

For limiting case $\mu_s N \ge f \Rightarrow \mu_s.Mg\cos\theta = \dfrac{2}{7}Mg\sin\theta \Rightarrow \mu_s \ge \dfrac{2}{7}\tan\theta$

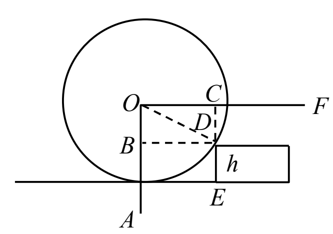
Example: A wheel of radius R and weight W is to be raised over an obstacle of height h by a horizontal force F applied to the centre as shown in figure. Find the minimum value of F



Solution: Taking torque about D, the corner of the obstacle,

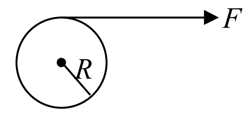
$$F \times CD = W \times BD$$

$$F = W\frac{BD}{CD} = W\sqrt{\frac{(OD)^2 - (OB)^2}{CE - DE}}$$



$$F = W\frac{BD}{CD} = W\sqrt{\frac{R^2 - (R-h)^2}{R-h}} = W\sqrt{\frac{2Rh - h^2}{R-h}}$$

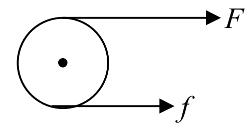
Example: A force F acts tangentially at the highest point of a sphere of mass M and radius R kept on a rough horizontal plane. If the sphere rolls without slipping, find the acceleration of center of sphere.



Solution: f is frictional force on the lowest point P

The force equation is equivalent to $F + f = Ma$ ....(i)

The torque equation is equivalent to $FR - fR = I_{cm}\alpha$

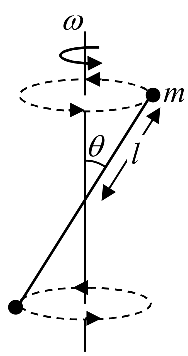


For case of rolling without slipping $\alpha = \frac{a}{R}$ and $I_{c.m} = \frac{2}{5}MR^2$

So torque equation can be reduce to $F - f = \frac{2}{5}Ma$ ....(ii)

So solving equation (i) and (ii) we will get $2F = \frac{7}{5}Ma \Rightarrow a = \frac{10F}{7M}$

Example: A thin mass less rod of length $2l$ has equal point masses m attached at its ends (see figure). The rod is rotating about an axis passing through its centre and making angle θ with it.



(a) Find the angular momentum of system

(b)Find the magnitude of the rate of change of its angular momentum i.e., $\left|\frac{d\vec{L}}{dt}\right|$

Solution: $I_{xx} = \sum_i m_i\left(y_i^2 + z_i^2\right) = 2ml^2$

$I_{yy} = \sum_i m_i\left(x_i^2 + z_i^2\right) = 0$, $I_{zz} = \sum_i m_i\left(y_i^2 + z_i^2\right) = 2ml^2$.

All component of product of inertia is zero, $I = \begin{bmatrix} 2ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2ml^2 \end{bmatrix}$ $\vec{\omega} = \begin{pmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$

$$\vec{L} = I\omega = \begin{bmatrix} 2ml^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2ml^2 \end{bmatrix}\begin{pmatrix} \omega\sin\theta \\ \omega\cos\theta \\ 0 \end{pmatrix} \Rightarrow \left(2ml^2\omega\sin\theta, 0, 0\right)$$

$$\left|\frac{d\vec{L}}{dt}\right| = \vec{\omega}\times L = 2ml^2\omega^2\sin\theta\cos\theta = ml^2\omega^2\sin 2\theta$$

Example: The linear mass density of a rod of length L varies from one end to the other as $\lambda_0\left(1+\frac{x^2}{L^2}\right)$, where x is the distance from one end with tensions T_1 and T_2 in them (see figure), and λ_0 is a constant. The rod is suspended from a ceiling by two massless strings.

(a) Find mass of Rod

(b) Find center of mass of Rod

(c) Find tension T_1 and T_2

Solution: (a) The mass of rod is $m=\int_0^L \lambda_0\left(1+\frac{x^2}{L^2}\right)dx=\frac{4\lambda_0 L}{3}$

(b) The centre of gravity of the rod is located at $x_{cm}=\frac{\int_0^L x\lambda_0\left(1+\frac{x^2}{L^2}\right)dx}{\int_0^L \lambda_0\left(1+\frac{x^2}{L^2}\right)dx}=\frac{9L}{16}$

(c) Force equation $T_1+T_2=\frac{4\lambda_0 Lg}{3}$

System is in equilibrium about center of mass so

From torque equation $T_1\times\frac{9L}{16}=T_2\times\left(L-\frac{9L}{16}\right)\Rightarrow T_1\times\frac{9L}{16}=T_2\times\frac{7L}{16}\Rightarrow T_2=T_1\times\frac{9}{7}$

By putting value of $T_2=\frac{9}{7}T_1$ in equation, we get $T_1+T_2=\frac{4\lambda_0 Lg}{3}$

$\frac{16}{7}T_1=\frac{4\lambda_0 Lg}{3}\Rightarrow T_1=\frac{7\lambda_0 Lg}{12}$ and $T_2=\frac{9}{7}\times T_1=\frac{9}{7}\times\frac{7\lambda_0 Lg}{12}=\frac{9\lambda_0 Lg}{12}$