Chapter 6

Linear Momentum And Energy

5. Energy Conservation Theorem

If there exists a scalar function $\phi(x,y,z,t)$, so that we could write $\vec{F}=\nabla\phi$. We shall say that the vector field $\vec{F}$ is a potential field. The scalar function $\phi(x,y,z,t)$ is then called the potential function of the field. The vector field $\vec{F}$ is called conservative if ϕ does not explicitly depend on time. The potential function $\phi(x,y,z)$, in this case, is called the force potential.

It is easy to show that if the force field is conservative the work done in moving the particle from 1 to 2 is independent of the path connecting 1 and 2. The total work done on the particle by the force $\vec{F}$ as it moves from 1 to 2 is given by $W_{12}=\int_1^2 \vec{F}\cdot d\vec{s}$

For a conservative force filed $W_{12}=\int_1^2 \vec{F}\cdot d\vec{s}=\int_1^2 \nabla\phi\cdot d\vec{s}=\int_1^2 \frac{d\phi}{ds}ds=\int_1^2 d\phi=\phi_2-\phi_1$

Thus, the total work done is equal to the difference in force potential no matter how the particle moves from 1 to 2 following differential relation $dW = \vec{F} \cdot d\vec{s} = d\phi$.

If we now write $\phi(x, y, z) = -U(x, y, z)$ (inserting a minus sign for reasons of convention) and express the force as $\vec{F} = -\nabla U$, then the scalar function U is called the potential energy of the particle.

When $\vec{F}$ is expressed as in the above equation, the work done becomes $W_{12} = U_1 - U_2$

It may be noted that the line integral of the field $\vec{F} = -\nabla U$ along a closed curve (called circulation) is zero as shown below $\oint_C \vec{F} \cdot d\vec{s} = -\oint_C dU = 0$

$$W_{12} = \int_1^2 dW = \int_1^2 dT \text{ it can be concluded that } T_1 + U_1 = T_2 + U_2 .$$

It says that the quantity $T + U$ remains a constant as the particle moves from point 1 to point 2.Since 1 and 2 are arbitrary points, we have obtained the statement of conservation of total mechanical energy $E = T + U =$ constant.

Thus, the energy conservation theorem states that the total energy of a particle in a conservative force field is constant. It is instructive to note that equation (6) does not uniquely determine the function ϕ. We could as well define $\vec{F} = \nabla\phi + c$, where c is any constant. Hence, the choice for the zero level of ϕ, and consequently U, is arbitrary.

We can verify directly from equation (11) that the total energy in a conservative field is a constant of the motion.

We have $\dfrac{dE}{dt} = \dfrac{dT}{dt} + \dfrac{dU}{dt}$.

The kinetic energy term can be written as $\dfrac{dT}{dt} = \dfrac{1}{2} m \dfrac{dv^2}{dt} = m \dfrac{d\vec{v}}{dtdt} = m \dfrac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v}$

The potential energy U depends on time only through the changing position of the particle: $U = U(\vec{s}(t)) = U(x(t), y(t), z(t))$. Thus, we have

$\dfrac{dU}{dt} = \dfrac{\partial U}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial U}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial U}{\partial z}\dfrac{dz}{dt} = \nabla U \cdot \vec{v} = -\vec{F} \cdot \vec{v}$. If follows that $\dfrac{dE}{dt} = \vec{F} \cdot \vec{v} - \vec{F} \cdot \vec{v} = 0$.

Thus, the total energy of the particle moving in a conservative force field is a constant during the motion.