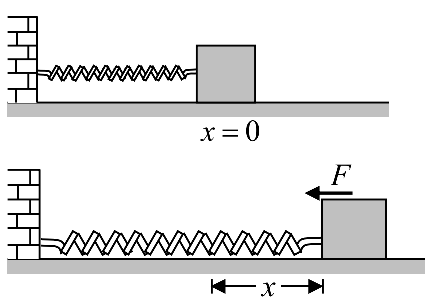
Chapter 6

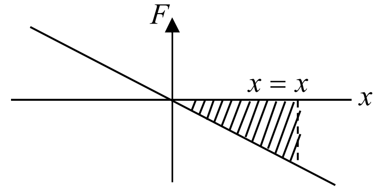
Linear Momentum And Energy

7. Spring Force

An important example of the above idea is a spring that obeys. Hooke's Law. Consider the situation shown in figure. One end of a spring is attached to a fixed vertical support and the other end to a block which can move on a horizontal table. Let $x = 0$ denote the position of the block when the spring is in its natural length.



When the block is displaced by an amount x (either compressed or elongated) a restoring force (F) is applied by the spring on the block. The direction of this force F is always towards its mean position ($x=0$) and the magnitude is directly proportional to x o



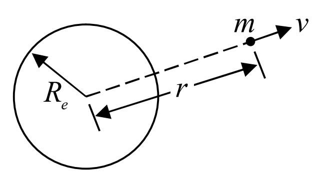
$$F \propto x \Rightarrow F = -kx = -\frac{dU}{dx} \Rightarrow U_2 - U_1 = \frac{1}{2}k\left(x_2^2 - x_1^2\right)$$

so potential energy of particle which is stored energy in spring is equivalent to $U = \frac{1}{2}kx^2$

Example: A mass m is shot vertically upward from the surface of the earth with initial speed v_0. Assuming that the only force is gravity, find its maximum altitude and the minimum value of v_0 for the mass to escape the earth completely.

Solution: The force on m is $F = -\frac{GM_e m}{r^2}$.

The problem is one dimensional in the variable r and it is simple to find the kinetic energy at distance r by the work-energy theorem.



Let the particle start at $r = R_e$ with initial velocity v_0.

$$K(r) - K(R_e) = \int_{R_e}^{r} F(r)\,dr = -GM_e m \int_{R_e}^{r} \frac{dr}{r^2}$$

$$\Rightarrow \frac{1}{2}m[v(r)]^2 - \frac{1}{2}mv_0^2 = GMm\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

We can immediately find the maximum height of m. At the highest point, $v(r) = 0$ and we have

$$v_0^2 = 2GM_e\left(\frac{1}{R_e} - \frac{1}{r_{\max}}\right).$$

It is a good idea to introduce known familiar constants whenever possible. For example

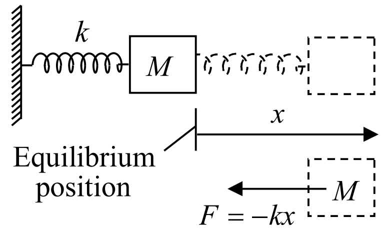
Since, $g = \frac{GM_e}{R_e^2}$, we can write, $v_0^2 = 2gR_e^2\left(\frac{1}{R_e} - \frac{1}{r_{\max}}\right) = 2gR_e\left(1 - \frac{R_e}{r_{\max}}\right)$ or $r_{\max} = \frac{R_e}{1 - \frac{v_0^2}{2gR_e}}$.

The escape velocity from the earth is the initial velocity needed to move $r_{\max}$ to infinity.

Then escape velocity, $v_{escape} = \sqrt{2gR_e} = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = 1.1 \times 10^4\, m/s$

Example: Consider a mass M attached to a spring with spring constant k. Using the coordinate x measured from the equilibrium point x_0 solve the Equation of motion for simple harmonic motion with the help of work energy theorem.

Solution: Consider a mass M attached to a spring.



Using the coordinate x measured from the equilibrium point, the spring force is $F=-kx$.

$$\frac{1}{2}Mv^2-\frac{1}{2}Mv_0^2=-k\int_{x_0}^{x}x\,dx=-\frac{1}{2}kx^2+\frac{1}{2}kx_0^2$$

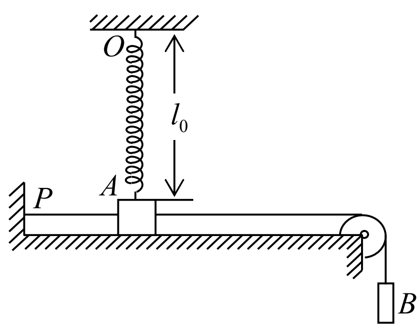
In order to find x and v, we must know their values at some time t_0. Let us consider the case where at $t=0$ the mass is released from rest, $v_0=0$, at a distance x_0 from the origin. Then

$$v^2=-\frac{k}{M}x^2+\frac{k}{M}x_0^2 \text{ and } \frac{dx}{dt}=\sqrt{\frac{k}{M}}\sqrt{x_0^2-x^2}$$

Separating the variables gives

$$\int_{x_0}^{x}\frac{dx}{\sqrt{x_0^2-x^2}}=\sqrt{\frac{k}{M}}\int_0^t dt=\sqrt{\frac{k}{M}}\,t\Rightarrow \sin^{-1}\frac{x}{x_0}=\sqrt{\frac{k}{M}}\,t$$

Example: A small bar A resting on a smooth horizontal plane is attached by threads to a point P and by means of a weightless pulley, to a weight B possessing the same mass as the bar itself. Besides the bar is also attached to a point O by means of a light non-deformed spring of length l_0 cm and stiffness $k=5mg/l_0$, where m is the mass of the bar. The thread PA have been burned, then the bar starts moving. Find its velocity at the moment when it is breaking off the plane.



Solution: When bar PA breaking off the plane, then normal reaction will be zero.

Force equation: $kl_0[\operatorname{cosec}\theta-1]\sin\theta=mg\Rightarrow kl_0[1-\sin\theta]=mg$

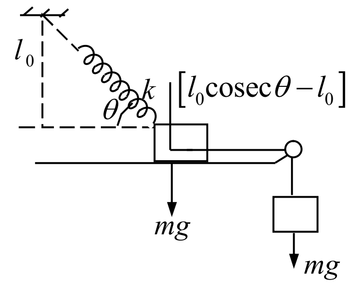
$$\Rightarrow \sin\theta=\frac{kl_0-mg}{k\ell_0}=1-\frac{mg}{kl_0} \quad \text{...(i)}$$

<u>Energy equation</u>: From work energy theorem,

Loss in P.E (mg) = Gain in K.E. of both block + spring energy.

$$mgl_0 \cot\theta = \left(\frac{1}{2}mv^2\right) + \left(\frac{1}{2}mv^2\right) + \frac{1}{2}kl_0^2(\operatorname{cosec}\theta - 1)^2 \qquad \text{...(ii)}$$

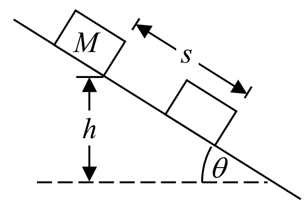
$\because k = 5mg/l_0$, $\sin\theta = \dfrac{kl_0 - mg}{kl_0} = \dfrac{4}{5}$, $\cos\theta = \dfrac{3}{5}$ and $\cot\theta = \dfrac{3}{4}$



from (i) and (ii): $\dfrac{3}{4}mgl_0 = mv^2 + \dfrac{1}{2}(5mgl_0)\left(\dfrac{5}{4} - 1\right)^2$

$$\Rightarrow \frac{3}{4}mgl_0 = mv^2 + \frac{5}{32}mgl_0 \Rightarrow v = \sqrt{\frac{19gl_0}{32}}$$

Example: A block of mass M slides down a plane of angle θ. The problem is to find the speed of the block after it has descended through height h, assuming that it starts from rest and that the coefficient of friction μ is constant.



(Initially the block is at rest at height h; finally the block is moving with speed v at height 0).

Solution: The potential energy, kinetic energy and total energy at point a and b respectively

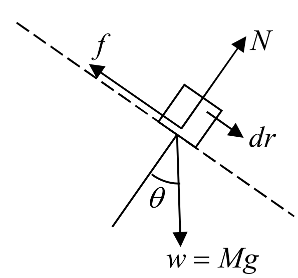
$$U_a = Mgh,\ U_b = 0\ \ K_a = 0\ \ , K_b = \frac{1}{2}Mv^2,\ E_a = Mgh\ \ , E_b = \frac{1}{2}Mv^2$$

The non-conservative force is $f = \mu N = \mu Mg\cos\theta$. Hence, conservative work is

$$W_{ba}^{nc} = \int_a^b f \cdot dr = -fs,$$

where s is the distance the block slides. The negative sign arises because the direction of f is always opposite to the displacement, so that $f \cdot dr = -f\,dr$. Using $s = h/\sin\theta$, we have

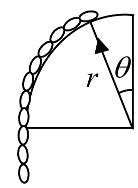
$$W_{ba}^{nc} = -\mu Mg\cos\theta\frac{h}{\sin\theta} = -\mu\cot\theta Mgh.$$



The energy equation $E_b - E_a = W_{ba}^{nc}$ becomes

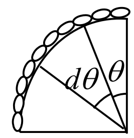
$\dfrac{1}{2}Mv^2 - Mgh = -\mu\cot\theta Mgh$, which gives $v = \left[2(1 - \mu\cot\theta)gh\right]^{1/2}$

Example: A chain of length $\frac{\pi r}{2}$, mass per unit length ρ is released from rest at $\theta = 0^0$. On a smooth surface. Find velocity of chain as it leaves the surface.



Solution: Taking a small element of chain taking angle $d\theta$ at point O, its mass $= rd\theta \times \rho$

P.E. of element $= dm \times g \times h = rd\theta \times \rho \times g \times r\cos\theta$



Total P.E. $= \int_{\theta=\sigma^0}^{\theta=\pi/2} rd\theta \times \rho g \times r\cos\theta$

$$U_1 = \rho g r^2$$

Final P.E. = mass of chain $\times g \times$ distance moved by centre of gravity of chain

$$U_2 = \frac{\pi r}{2}\rho g \times \left(\frac{-\pi r/2}{2}\right) \text{ or } U_2 = -\frac{\pi^2 r^2 \rho g}{8} \Delta U = \rho g r^2\left(1 + \frac{\pi}{8}\right)$$

or gross in K.E. = Loss in P.E.

$$\Rightarrow \frac{1}{2} \times \left(\frac{\pi r}{2} \times \rho\right) v^2 = \rho g r^2 \left(1 + \frac{\pi^2}{8}\right) \Rightarrow v = \sqrt{gr\left(\frac{\pi}{2} + \frac{4}{\pi}\right)}$$