Chapter 6

Linear Momentum And Energy

8. Conservation of Momentum and Collision

In the last section we found that the total external force F acting on a system is related to the total momentum P of the system by $F = \frac{dP}{dt}$

Consider the implications of this for an isolated system, that is, a system which does not interact with its surroundings. In this case $F = 0$, and $dP/dt = 0$. The total momentum is constant; no matter how strong the interactions among an isolated system of particles, and no matter how complicated the motions, the total momentum of an isolated system is constant. This is the law of conservation of momentum. As we shall show, this apparently simple law can provide powerful insights into complicated systems.

Collisions

Contrary to the meaning of the term 'collision' in our everyday life in Physics it does not necessarily mean one particle 'striking' against other. Indeed two particles may not even touch each other and may still be said to collide. All that is implied is that as the particles approach each other.

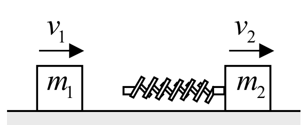
(i) An impulse (a large for a relatively short time) acts on each colliding particles.

(ii) the total momentum of the particles remain conserved.

The collision is in fact a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensible in dealing with the phenomenon of collision between particles. Consider a situation shown in figure.

Two blocks of masses m_1 and m_2 are moving with velocities v_1 and $v_2\left(<v_1\right)$ along the same straight line in a smooth horizontal surface. A spring is attached to the block of mass m_2. Now, let us see what happens during the collision between two particles.

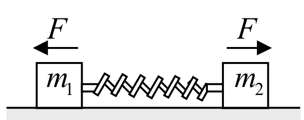
Figure (a) Block of mass m_1 is behind m_2 since $v_1>v_2$, the blocks will collide after some time.



$v_1>v_2$

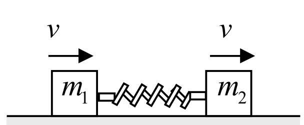
(a)

Figure (b) The spring is compressed. The spring force $F(=kx)$ acts on the two blocks in the directions shown in figure. This force decreases the velocity of m_1 and increases the velocity of m_2.



(b)

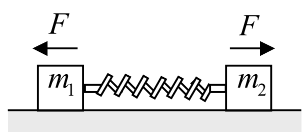
Figure (c) The spring will compress till velocity of both the blocks become equal. So, at maximum compression (say x_m) velocities of both the blocks are equal (say v)



Maximum compression $=x_m$

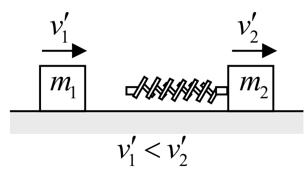
(c)

Figure (d) Spring force is still in the directions shown in figure, i.e., velocity of block m_1 is further decreased and that of m_2 is increased. The spring now starts relaxing.



(d)

Figure (e) The two blocks are separated from one another. Velocity of block m_2 becomes more than the velocity of block m_1, i.e. $v_2' > v_1'$.

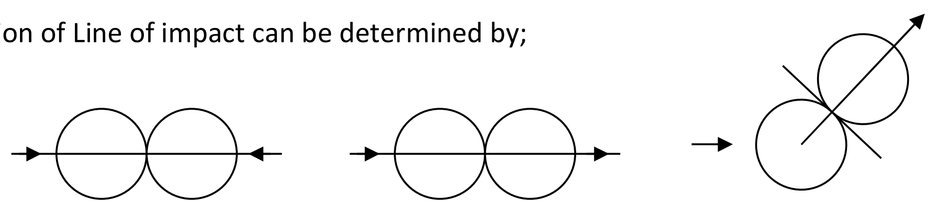


(e)

Line of Impact

The line passing through the common normal to the surfaces in contact during impact is called line of impact. The force during collision acts along this line on both the bodies.

Direction of Line of impact can be determined by;



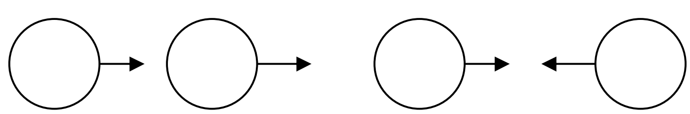
(a) Geometry of colliding objects like spheres, discs, wedge etc.

(b) Direction of change of momentum.

If one particle is stationary before the collision, then the line of impact will be along its motion after collision.

Classification of Collisions

(A) Head-on collision: If the velocities of the particles are along the same line before and after the collision,



(i) Elastic collision: In an elastic collision, the particle regain their shape and size completely after collision. i.e., no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus, kinetic energy of system after collision is equal to kinetic energy of system before collision. Thus, in addition to the linear momentum, kinetic energy also remains conserved before and after collision.

Consider two elastic bodies A and B moving along the same line (figure). The body A has a mass m_1 and moves with a velocity v_1 towards right and the body B has a mass m_2 and moves with a velocity v_2 in the same direction. We assume $v_1 > v_2$ so that the two bodies may collide. Let v_1' and v_2' be the final velocities of the bodies after the collision. The total linear momentum of the two bodies remains constant, so that,

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \quad \text{(i)}$$

or $$m_1v_1 - m_1v_1' = m_2v_2' - m_2v_2$$

or $$m_1\left(v_1 - v_1'\right) = m_2\left(v_2' - v_2\right) \quad \text{(ii)}$$

Also, since the collision is elastic, the kinetic energy before the collision is equal to the kinetic energy after the collision. Hence,

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$$

or, $$m_1v_1^2 - m_1v_1'^2 = m_2v_2'^2 - m_2v_2^2$$

or, $$m_1\left(v_1^2 - v_1'^2\right) = m_2\left(v_2'^2 - v_2^2\right) \quad \text{(iii)}$$

Dividing (iii) by (ii)

$$v_1 + v_1' = v_2' + v_2$$

$$v_1 - v_2 = v_2' - v_1' \quad \text{(iv)}$$

Now, $\left(v_1 - v_2\right)$ is the rate at which the separation between the bodies decreases before the collision. Similarly, $\left(v_2' - v_1'\right)$ is the rate of increase of separation after the collision. So the equation (iv) may be written as

Velocity of separation (after collision) = Velocity of approach (before collision)

This result is very useful in solving problems involving elastic collision. The final velocities v_1' and v_2' may be obtained from equation (i) and (iv). Multiply equation (iv) by m_2 and subtract from equation (i).

$$2m_2v_2 + (m_1 - m_2)v_1 = (m_1 + m_2)v_1'$$

or $$v_1' = \frac{(m_1 - m_2)}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2 \qquad \text{(A)}$$

Now multiply equation (iv) by m_1 and add to equation (i),

$$2m_1v_1 - (m_1 - m_2)v_2 = (m_2 + m_1)v_2'$$

or $$v_2' = \frac{2m_1v_1}{m_1 + m_2} - \frac{(m_1 - m_2)v_2}{m_1 + m_2} \qquad \text{(B)}$$

Equations, (A) and (B) give the final velocities in terms of the initial velocities and the masses.

For inelastic collision velocities after collision are

$$v_1' = \frac{(m_1 - em_2)}{m_1 + em_2}v_1 + \frac{(m_2 + em_2)}{m_1 + em_2}v_2$$

$$v_2' = \frac{(m_1 + em_1)}{m_1 + m_2}v_1 + \frac{(m_2 - em_1)}{m_1 + m_2}v_2$$

Special Cases

(a) Elastic collision between a heavy body and a light body:

Let $m_1 >> m_2$. A heavy body hits a light body from behind.

We have, $\frac{m_1 - m_2}{m_1 + m_2} \approx 1$, $\frac{2m_2}{m_1 + m_2} \approx 0$ and $\frac{2m_1}{m_1 + m_2} \approx 2$

With these approximations the final velocities of the bodies are, from (A) and (B),

$v_1' \approx v_1$ and $v_2' \approx 2v_1 - v_2$.

The heavier body continues to move with almost the same velocity, If the lighter body were kept at rest $v_2 = 0, v_2' = 2v_1$ which means the lighter body, after getting a push from the heavier body will fly away with a velocity double the velocity of the heavier body.

Next, suppose $m_2 >> m_1$. A light body hits a heavy body from behind.

We have,

$$\frac{m_1 - m_2}{m_1 + m_2} \approx -1, \quad \frac{2m_2}{m_1 + m_2} \approx 2 \text{ and } \frac{2m_1}{m_1 + m_2} \approx 0$$

The final velocities of the bodies are, from (A) and (B), $v_1' \approx -v_1 + 2v_2$ and $v_2' \approx v_2$

The heavier body continues to move with almost the same velocity, the velocity of the lighter body changes. If the heavier body were at rest, $v_2 = 0$ then $v_1' = -v_1$ the lighter body returns alter collision with almost the same speed. This is the case when a ball collides elastically with a fixed wall and returns with the same speed.

(b) Elastic collision of two bodies of equal mass

Putting $m_1 = m_2$ in equation (A) and (B) $v_1' = v_2$ and $v_2' = v_1$.

When two bodies of equal mass collide elastically, their velocities are mutually interchanged.

Theorem: *In a 1-D elastic collision, the relative velocity of the two particles after the collision is the negative of the relative velocity before the collision.*

Proof: Let the masses be m and M. Let v_i and V_i be the initial velocities, and let v_f and V_f be the final velocities. Conservation of momentum and energy give

$$mv_i + MV_i = mv_f + MV_f$$
$$\frac{1}{2}mv_i^2 + \frac{1}{2}MV_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV_f^2 \qquad \text{(A)}$$

Rearranging these yields

$$m\left(v_i - v_f\right) = M\left(V_f - V_i\right)$$
$$m\left(v_i^2 - v_f^2\right) = M\left(V_f^2 - V_i^2\right) \qquad \text{(B)}$$

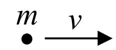
Dividing the second equation by the first gives $v_i + v_f = V_i + V_f$. Therefore,

$$v_i - V_i = -\left(v_f - V_f\right)$$

as we wanted to show. In taking the quotient of these two equations, we have lost the $v_f = v_i$ and $V_f = V_i$ solution. But as stated in the above example, this is the trivial solution.

This is a splendid theorem. It has the quadratic energy-conservation statement built into it. Hence, using this theorem along with momentum conservation (both of which are linear equations and thus easy to deal with) gives the same information as the standard combination of Eqs. (A). Another quick proof is the following. It is fairly easy to see that the theorem is true in the CM frame, so it is therefore true in any frame, because it involves only differences in velocities.

Example: (Two masses in 1-D, again): A mass *m* with speed v approaches a stationary mass M as given in figure below. The masses bounce off each other elastically. What are the final velocities of the particles? Assume that the motion takes place in 1-D.



Solution: Let v_1 and v_2 be the final velocities of the masses. Then conservation of momentum and energy give, respectively,

$$mv + 0 = mv_1 + Mv_2$$

$$\frac{1}{2}mv^2 + 0 = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 \qquad \text{(a)}$$

We must solve these two equations for the two unknowns v_1 and v_2. Solving for v_2 in the first equation and substituting into the second gives

$$mv^2 = mv_1^2 + M\frac{m^2\left(v - v_1\right)^2}{M^2}$$

$$\Rightarrow 0 = \left(m + M\right)v_1^2 - 2mvv_1 + \left(m - M\right)v^2$$

$$\Rightarrow 0 = \left(\left(m + M\right)v_1 - \left(m - M\right)v\right)\left(v_1 - v\right)$$

One solution is $v_1 = v$, but this isn't the one we're concerned with. It is of course a solution, because the initial conditions certainly satisfy conservation of energy and momentum with the initial condition (a fine tautology indeed). If you want, you can view $v_1 = v$ as the solution

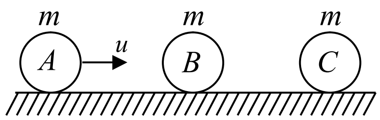
where the particles miss each other. The fact that $v_1 = v$ is always a root can often save you a lot of quadratic-formula trouble.

The $v_1 = v\left(m - M\right)/\left(m + M\right)$ root is the one we want. Plugging this v_1 back into the first of equation (a) to obtain v_2 gives

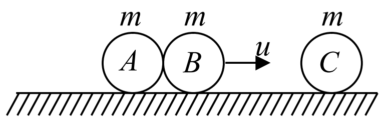
$$v_1 = \frac{\left(m - M\right)v}{m + M} \text{ and } v_2 = \frac{2mv}{m + M} \qquad \text{(b)}$$

This solution was somewhat of a pain, because it involved a quadratic equation. The following theorem is extremely useful because it offers a way to avoid the hassle of quadratic equations when dealing with 1-D elastic collisions.

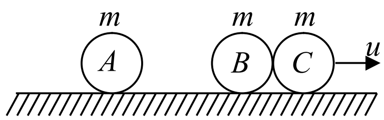
Example: Three balls A, B and C of same mass 'm' are placed on a frictionless horizontal plane in a straight line as shown. Ball A is moved with velocity u towards the middle ball B. If all the collisions are elastic then, find the final velocities of all the balls.



Solution: A collides elastically with B and comes to rest but B starts moving with velocity u



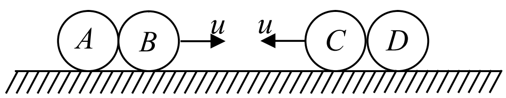
After a white B collides elastically with C and comes to rest but C starts moving with velocity u.



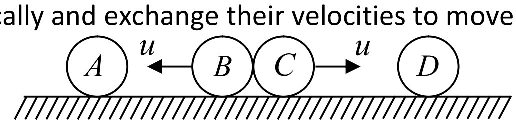
$\therefore$ Final velocities $V_A = 0; V_B = 0$ and $V_C = u$

Example: Four identical balls A, B, C and D are placed in a line on a frictionless horizontal surface. A and D are moved with same speed 'u' towards the middle as shown. Assuming elastic collisions, find the final velocities.

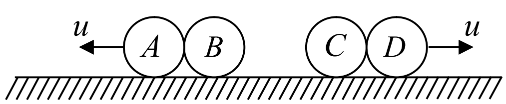
Solution: A and D collides elastically with B and C respectively and come to rest but B and C starts moving with velocity u towards each other as shown.



B and C collides elastically and exchange their velocities to move in opposite, directions

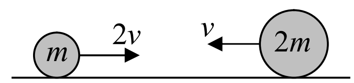


Now; B and C collides elastically with A and D respectively and come to rest but A and D starts moving with velocity u away from each other as shown



$\therefore$ Final velocities $V_A = u(\leftarrow) : V_B = 0; V_C = 0$ and $V_D = u(\rightarrow)$

Example: Two particles of mass m and $2m$ moving in opposite directions Collide elastically with velocity v and $2v$ respectively. Find their velocities after collision.



Solution: Let the final velocities of m and $2m$ be v_1 and v_2 respectively as shown in the figure

By conservation of momentum:

$m(2v)+2m(-v)=m(v_1)+2m(v_2)$ or $0=mv_1+2mv_2$

or $\quad v_1+2v_2=0 \quad$ (i)

and since the collision is elastic:

$v_2-v_1=2v(-v)$ or $v_2-v_1=3v \quad$ (ii)

Solving the above two equations, we get,

$v_2=v$ and $v_1=-2v$

i.e., the mass $2m$ returns with velocity v while the mass m returns with velocity $2v$ in the direction in the figure:

(ii) Inelastic collision: In an inelastic collision, the particle do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved. However, in the absence of external forces, law of conservation of linear momentum still holds good. Practically most of the collisions are inelastic.

Example: A block m_1 strikes a stationary block m_3 perfectly inelastically in a head on collision. Another block m_2 is kept on m_3 Neglecting the friction between all contacting surfaces. Find the fractional decrease of K.E of the system in collision.

Solution: Since the impact between m_1 and m_3, is inelastic, m_1, and m_3, will move together towards right but m_2 does not move due to the absence of friction.

The velocity of the combined mass, $v'=\dfrac{m_1 v}{m_1+m_2+m_3}$

$$\frac{|\Delta K.E|}{K.E}=\frac{\frac{1}{2}m_1v^2-\frac{1}{2}(m_1+m_3)v'^2}{\frac{1}{2}m_1v^2}=1-\left(\frac{m_1+m_3}{m_1}\right)\left(\frac{v'}{v}\right)^2$$

$$=1-\left(\frac{m_1+m_3}{m_1}\right)\left(\frac{m_1}{m_1+m_3}\right)^2=1-\frac{m_1}{m_1+m_2}=\frac{m_3}{m_1+m}$$

(iii) Perfectly inelastic or perfectly plastic ($e=0$): If velocity of separation just after collision becomes zero then the collision is perfectly inelastic. Collision is said to be perfectly inelastic if both the particles stick together after collision and move with same velocity,

Note: Actually, collision between all real objects are neither perfectly elastic nor perfectly inelastic, its inelastic in nature.