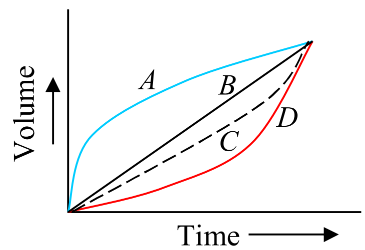
NET-JRF JUNE 2021 [SOLUTION]

PART A

Q1. An inverted cone is filled with water at a constant rate. The volume of water inside the cone as a function of time is represented by the curve



(a) A (b) B (c) C (d) D

Ans.: (b)

Q2. A spacecraft flies at a constant height R above a planet of radius R. At the instant the spacecraft is over the north pole, the lowest latitude visible from the spacecraft is:

(a) 0^0 (equator) (b) $30^0 N$ (c) $45^0 N$ (d) $60^0 N$

Ans. : (b)

Q3. An experiment consists of tossing a coin 20 times. Such an experiment is performed 50 times. The number of heads and the number of tails in each experiment are noted. What is the correlation coefficient between the two?

(a) -1 (b) $-20/50$ (c) $20/50$ (d) 1

Ans. : (a)

Q4. Which of these groups of numbers has the smallest mean?

Group A: $1,2,3,4,5,6,7,8,9$

Group B: $1,2,3,4,6,6,7,8,9$

Group C: $1,2,2,4,5,6,7,8,9$

Group D: $1,3,3,4,5,6,7,9,9$

(a) A (b) B (c) C (d) D

Ans. : (c)

Q5. Identical balls are tightly arranged in the shape of an equilateral triangle with each side containing n balls. How many balls are there in the arrangement?

(a) $\frac{n^2}{2}$ (b) $\frac{n(n+1)}{2}$ (c) $\frac{n(n-1)}{2}$ (d) $\frac{(n+1)^2}{2}$

Ans. : (b)

Q6. A shopkeeper has a faulty pan balance with a zero offset. When an object is placed in the left pan it is balanced by a standard $100\ g$ weight. When it is placed in the right pan it is balanced by a standard $80\,g$ weight. What is the actual weight of the object?

(a) $90\,g$ (b) $88.88\,g$ (c) $95\ g$ (d) $85\ g$

Ans. : (a)

Q7. A and B start from the same point in opposite directions along a circular track simultaneously. Speed of B is $2/3^{rd}$ that of A. How many times will A and B cross each other before meeting at the starting point?

(a) 2 (b) 3 (c) 5 (d) 4

Ans. : (d)

Q8. Consider a solid cube of side 5 units. After painting, it is cut into cubes of 1 unit. Find the probability that a randomly chosen unit cube has only one side painted.

(a) $\frac{56}{125}$ (b) $\frac{36}{125}$ (c) $\frac{44}{125}$ (d) $\frac{54}{125}$

Ans. : (d)

Q9. How many integers in the set $\{1,2,3,....,100\}$ have exactly 3 divisors?

(a) 4 (b) 12 (c) 5 (d) 9

Ans. : (a)

Q10. The arithmetic and geometric means of two numbers are 65 and 25, respectively. What are these two numbers?

(a) $110,20$ (b) $115,15$ (c) $120,10$ (d) $125,5$

Ans. (d)

Q11. Shyam spent half of his money and was left with as many as he had rupees before, but with half as many rupees as he had paise before. Which of the following is a possible amount of money he is left with?

(a) 49 rupees and 98 paise (b) 49 rupees and 99 paise

(c) 99 rupees and 99 paise (d) 99 rupees and 98 paise

Ans. : (b)

Q12. A cylindrical road roller having a diameter of $1.5\,m$ moves at a speed of $3\ km/h$ while levelling a road. How much length of the road will be levelled in 45 minutes?

(a) $2.25\ km$ (b) $0.375\ \pi km$ (c) $0.75\ \pi km$ (d) $1.5\ km$

Ans. : (a)

Q13. An intravenous fluid is given to a child of $7.5\ kg$ at the rate of 20 drop/minute. The prescribed dose of the fluid is $40\,\text{ml}$ per kg of body weight. If the volume of a drop is $0.05\ \text{ml}$, how many hours are needed to complete the dose?

(a) 2 (b) 3 (c) 4 (d) 5

Ans. : (d)

Q14. A cousin is a non-sibling with a common ancestor. If there is exactly one pair of siblings in a group of 5 persons then the maximum possible number of pairs of cousins in the group is

(a) 3 (b) 6 (c) 9 (d) 10

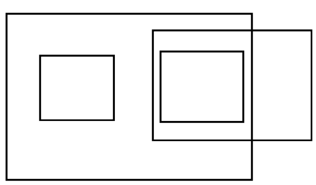
Ans. : (c)

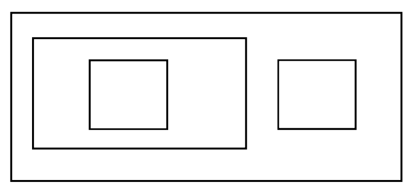
Q15. In a tournament with 8 teams, a win fetches 3 points and a draw, 1. After all terms have played three matches each, total number of points earned by all teams put together must lie between

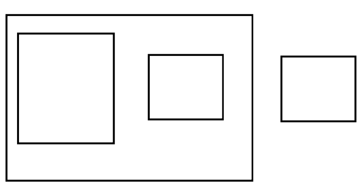
(a) 24 and 36 (b) 24 and 32 (c) 12 and 24 (d) 32 and 48

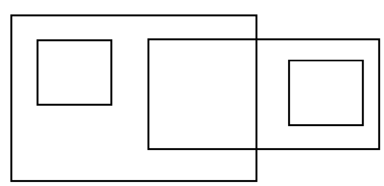
Ans. : (a)

Q16. An appropriate diagram to represent the relations between the categories KEYBOARD, HARDWARE, OPERATING SYSTEM and CPU is

(a) 

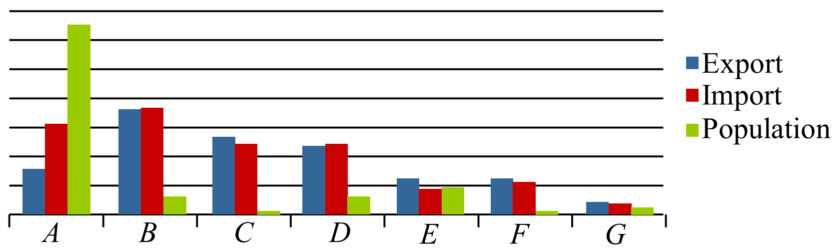
(b) 

(c) 

(d) 

Ans. : (c)

Q17. Trade figures and populations in appropriate units in a certain year are given for 7 countries.



If countries are ranked according to the difference in their per capita exports over import, then the best and worst ranking countries are respectively

(a) C and A (b) A and E (c) C and B (d) A and F

Ans. : (a)

Q18. At least two among three persons A, B and C are truthful. If A calls B a liar and if B calls C a liar, then which of the following is FALSE?

(a) A is truthful (b) B is truthful

(c) C is truthful (d) At least one is a liar

Ans. : (b)

Q19. The maximum area of a right-angled triangle inscribed in a circle of radius r is

(a) $2r^2$ (b) $\frac{r^2}{2}$ (c) $\sqrt{2}r^2$ (d) r^2

Ans. : (d)

Q20. If we replace the mathematical operations in the expression $(11+4+2)+24\times6$ as given in the table:

Operation	$+$	$-$	$\times$	$\div$
Replaced by	$-$	$\times$	$\div$	$+$

Then its new value is

(a) $\frac{23}{6}$ (b) 1 (c) 18 (d) 7

Ans. : (d)

PART B

Q1. A particle in one dimension executes oscillatory motion in a potential $V(x)=A|x|$, where $A>0$ is a constant of appropriate dimension. If the time period T of its oscillation depends on the total energy E as E^{α}, then the value of α is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

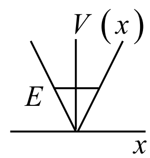
Topic: Classical Mechanics

Sub Topic: Action Angle

Ans. : (b)

Solution: $V(x)=A|x|$

$$J=\oint\sqrt{E-V(x)}dx=4\int_0^{E/A}\sqrt{2m(E-Ax)}dx$$



$$H=\frac{P^2}{2m}+A|x|$$

$$J=\oint\sqrt{2m(E-A|x|)}dx=4\int_0^{E/A}\sqrt{2m}\sqrt{E-Ax}dx$$

Let, $u=E-Ax \Rightarrow du=-Adx$

$$J=4\int_E^0\sqrt{2m}u^{1/2}du=\frac{8\sqrt{2m}}{3A}E^{3/2}\quad\Rightarrow T=\frac{\partial J}{\partial E}\quad\Rightarrow\frac{4\sqrt{2mE}}{A}=\frac{\sqrt{32mE}}{A}\quad T\propto E^{1/2}$$

Q2. The equation of motion of a one-dimensional forced harmonic oscillator in the presence of a dissipative force is described by $\frac{d^2x}{dt^2}+10\frac{dx}{dt}+16x=6te^{-8t}+4t^2e^{-2t}$. The general form of the particular solution, in terms of constants A,B etc, is

(a) $t\left(At^2+Bt+C\right)e^{-2t}+\left(Dt+E\right)e^{-8t}$ (b) $\left(At^2+Bt+C\right)e^{-2t}+\left(Dt+E\right)e^{-8t}$

(c) $t\left(At^2+Bt+C\right)e^{-2t}+t\left(Dt+E\right)e^{-8t}$ (d) $\left(At^2+Bt+C\right)e^{-2t}+t\left(Dt+E\right)e^{-8t}$

Topic: Mathematical Physics

Sub Topic: Differential Equation

Ans. : (c)

Solution: $\frac{d^2x}{dt^2}+10\frac{dx}{dt}+16x=6te^{-Bt}+4t^2e^{-2t}$

$$x=C.F.+P.I.$$

$$D^2+10D+16=0$$

$D^2+2D+8D+16=0$

$D(D+2)+8(D+2)=0,2,-8$

$C.F.=C_1e^{-2t}+C_2e^{-8t}$

$P.I.\to \frac{1}{(D+2)(D+8)}\left(6te^{-8t}+4te^{-2t}\right)$

$\frac{1}{(D+2)(D+8)}=\frac{A}{D+2}+\frac{B}{D+8}=\frac{A(D+8)+B(D+2)}{(D+2)(D+8)}$

$1=(A+B)D+8A+2B$

$A+B=0,\ 8A+2B=1$

$A=-B,\quad 8A-2A=1$

$A=\frac{1}{6}$

$\left[\frac{1/6}{D+2}+\frac{-1/6}{D+8}\right]\left(6te^{-8t}+4t^2e^{-2t}\right)=\frac{1}{D+2}te^{-8t}+\frac{2}{3}\frac{1}{D+2}t^2e^{-2t}-\frac{1}{D+8}te^{-8t}-\frac{2}{3}\frac{1}{D+8}t^2e^{-2t}$

$=e^{-8t}\frac{1}{D-6}t+\frac{2}{3}e^{-2t}\frac{1}{D}t^2-e^{-8t}\frac{1}{D}t-\frac{2e^{-2t}}{3}\frac{1}{D+6}t^2$

$=-\frac{e^{-8t}}{6}\left[1-\frac{D}{6}\right]^{-1}t+\frac{2}{3}e^{-2t}\frac{t^3}{3}-e^{-8t}\frac{t^2}{2}-\frac{e^{-2t}}{9}\left[1+\frac{D}{6}\right]^{-1}t^2$

$=-\frac{e^{-8t}}{6}\left[t+\frac{1}{6}\right]+\frac{2}{9}t^3e^{-2t}-\frac{1}{2}t^2e^{-8t}-\frac{e^{-2t}}{9}\left[1-\frac{D}{6}+\frac{D^2}{36}\right]t^2$

$=-\frac{e^{-8t}}{6}\left[t+\frac{1}{6}\right]+\frac{2}{9}t^3e^{-2t}-\frac{1}{2}t^2e^{-8t}-e^{2t}\left(\frac{t^2}{9}-\frac{1}{27}t+\frac{1}{12}\right)$

$x=C.F.+P.I.$

$=C_1e^{-2x}+C_2e^{-8x}-\frac{1}{2}e^{-8t}t^2-\frac{1}{6}e^{-8t}t+\frac{2}{9}t^3e^{-2t}-\frac{1}{9}t^2e^{-2t}+\frac{te^{-2t}}{27}$

$P.I.=t\left(At^2+Bt+C\right)e^{-2t}+t\left(Dt+E\right)e^{-8t}$

Where, $y_f=\frac{1}{f(x)}e^{ax}V=e^{ax}\frac{1}{f(D+a)}V$

Q3. The vector potential for an almost point like magnetic dipole located at the origin is $A=\frac{\mu\sin\theta}{4\pi r^2}\hat{\phi}$, where (r,θ,ϕ) denote the spherical polar coordinates and $\hat{\phi}$ is the unit vector along ϕ. A particle of mass m and charge q, moving in the equatorial plane of the dipole, starts at time $=t=0$ with an initial speed v_0 and an impact parameter b. Its instantaneous speed at the point of closest approach is

(a) v_0 (b) $0/0$ (c) $v_0+\frac{\mu q}{4\pi mb^2}$ (d) $\sqrt{v_0^2+\left(\frac{\mu q}{4\pi mb^2}\right)^2}$

Topic: Electromagnetic Theory

Sub Topic: Magnetic dipole

Ans. : (a)

Solution: Initial speed is v_0, $A=\frac{\mu\sin(\theta)}{4\pi r^2}\hat{\phi}$

Let us consider the speed at closet approach is v_c

According to work energy principle

Work done =change in kinetic energy $=\frac{1}{2}mv_c^2-\frac{1}{2}mv_0^2=\int\vec{F}.\vec{dr}$

Since, magnetic force do not perform any work

$\int\vec{F}.\vec{dr}=0$

$\frac{1}{2}mv_c^2-\frac{1}{2}mv_0^2=\int\vec{F}.\vec{dr}=0\Rightarrow\frac{1}{2}mv_c^2-\frac{1}{2}mv_0^2=0\Rightarrow v_0=v_c$

Q4. A particle, thrown with a sped v from the earth's surface, attains a maximum height h (measured from the surface of the earth). If v is half the escape velocity and R denotes the radius of earth, then $\frac{h}{R}$ is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

Topic: Classical Mechanics

Sub Topic: Gravitation

Ans. : (b)

Solution: $\frac{1}{2}mv^2 - \frac{GmM}{R} = -\frac{GmM}{R+h}$ (i)

$v_e = \sqrt{\frac{2GM}{R}}$ and given that $r = \frac{1}{2}v_e$

From (i), $\frac{1}{2}m \times \frac{1}{4} \times \frac{2GmM}{R} - \frac{GmM}{R} = -\frac{GmM}{R+h}$

$\Rightarrow -\frac{3}{4}\frac{GmM}{R} = -\frac{GmM}{R+h} \Rightarrow 3R + 3h = 4R \Rightarrow \frac{h}{R} = \frac{1}{3}$

Q5. A particle of mass $\frac{1GeV}{c^2}$ and its antiparticle, both moving with the same speed v, produce a new particle X of mass $\frac{10\,GeV}{c^2}$ in a head-on collision. The minimum value of v required for this process is closest to

(a) $0.83c$ (b) $0.93c$ (c) $0.98c$ (d) $0.88c$

Topic: Classical Mechanics

Sub Topic: STR

Ans. : (c)

Solution: 0.98c topic classical mechanics STR (relativistic mass)

From conservation of momentum particle will rest after collision.

$$\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}=Mc^2$$

$$\frac{2m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}=Mc^2 \Rightarrow \frac{2\frac{1GeV}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{10GeV}{c^2} \Rightarrow \sqrt{1-\frac{v^2}{c^2}}=\frac{1}{5}$$

$$v=\sqrt{\frac{24}{25}}c=0.98c$$

Q6. The volume of the region common to the interiors of two infinitely long cylinders defined by $x^2+y^2=25$ and $x^2+4z^2=25$ is best approximated by

(a) 225 (b) 333 (c) 423 (d) 625

Topic: Mathematical Physics

Sub Topic: Vector Analysis

Ans. : (b)

Solution: $x^2+y^2=25,\ x^2+4z^2=25$

Common interior volume of two infinitely long cylinder will be 333

Q7. The volume integral

$$I=\iiint_V A\cdot(\nabla\times A)\,d^3x$$

is over a region V bounded by a surface Σ (an infinitesimal area element being $\hat{\mathbf{n}}ds$, where $\hat{\mathbf{n}}$ is the outward unit normal). If it changes to $I+\Delta I$, when the vector $\mathbf{A}$ is changed to $\mathbf{A}+\nabla\Lambda$, then ΔI can be expressed as

(a) $\iiint_V \nabla.(\nabla\Lambda\times A)\,d^3x$

(b) $-\iiint_V \nabla^2\Lambda\, d^3x$

(c) $-\oiint_\Sigma (\nabla\Lambda\times\mathbf{A}).\hat{\mathbf{n}}ds$

(d) $\oiint_z \nabla\Lambda.\,\hat{\mathbf{n}}ds$

Topic: Mathematical Physics

Sub Topic: Vector Analysis

Ans. : (c)

Solution: $I=\iiint A.(\nabla\times A)d^3v$

$I+\Delta I \Rightarrow A+\Delta A$

$I+\Delta I=\iiint (A+\Delta A)\cdot\nabla\times(A+\Delta A)\,d^3v$

$=\iiint A.(\nabla\times A)d^3v+\iiint \Delta A.(\nabla\times A)d^3v \qquad [\text{Since}, \nabla\times\Delta A=0]$

$=I+\iiint \Delta A.(\nabla\times A)d^3v$

$\Delta I=\iiint \Delta A\cdot(\nabla\times A)d^3v=\iiint \Delta A\cdot(\nabla\times A)d^3v \quad =\iiint \nabla\cdot(A\times\Delta A)d^3v \quad [\text{Since}, A\cdot B\times C=B\cdot C\times A]$

$=\iiint \nabla.(A\times\Delta A)d^3v=-\iiint \nabla.(\Delta A\times A)d^3v=-\oiint(\Delta A\times A)\hat{n}ds \ [\text{Since } \vec{A}\times\vec{B}=-\vec{B}\times\vec{A}]$

Q8. A generic 3×3 real matrix A has eigenvalues $0, 1$ and 6 and I is the 3×3 identity matrix. The quantity/quantities that cannot be determined from this information is/are the

(a) eigenvalues $(I+A)^{-1}$

(b) eigenvalues of $(I+A^TA)$

(c) determinant of A^TA

(d) rank of A

Topic: Mathematical Physics

Sub Topic: Matrices

Ans. : (b)

Solution: (a) eigenvalues $(I+A)^{-1}$: $A+I=B$ hence we know the eigenvalue of B, we can easily find the eigenvalue of B^{-1}.

(b) eigenvalues of $(I+A^TA)$: We don't know that what is the matrix A^TA, so eigenvalue **can not be** determined

(c) $|A^TA|=|A^T||A|=0$, product of eigenvalues of A^T and A.

(d) rank of A, hence eigenvalues are non-degenerate so matrices can be diagonalized and one eigenvalue is zero, so rank will be $3-1=2$.

Q9. A discrete random variable X takes a value from the set $\{-1,0,1,2\}$ with the corresponding probabilities $p(X)=\frac{3}{10},\frac{2}{10},\frac{2}{10}$ and $\frac{3}{10}$, respectively. The probability distribution $q(Y)=(q(0),q(1),q(4))$ of the random variable $Y=X^2$ is

(a) $\left(\frac{1}{5},\frac{3}{5},\frac{1}{5}\right)$ (b) $\left(\frac{1}{5},\frac{1}{2},\frac{3}{10}\right)$ (c) $\left(\frac{2}{5},\frac{2}{5},\frac{1}{5}\right)$ (d) $\left(\frac{3}{10},\frac{3}{10},\frac{2}{5}\right)$

Topic: Mathematical Physics

Sub Topic: Probability

Ans. : (b)

Solution: $X=[-1,0,1,2]$ $p(X)=3/10,\ 3/10,\ 2/10,\ 3/10$

$q(y)=[q(0),q(1),q(4)]\ ,y=x^2$

$q(y)=[q(0),q(1),q(4)]\ ,y=x^2$

$P(y=0)=P(x=0)=2/10\ [\text{sinec}, y=x^2]$

$P(y=1)=P(x=1)+P(x=-1)=3/10+2/10=5/10=1/2\ [\text{sinec}, y=x^2]$

$P(y=0)+P(y=1)+P(y=4)=1 \Rightarrow P(y=4)=1-2/10-1/2=1-7/10=3/10$

Thus, the correct probability distribution is $2/10$, $1/2$, $3/10$

Q10. The components of the electric field, in a region of space devoid of any charge or current sources, are given to be $E_i=a_i+\sum_{j=1,2,3} b_{ij}x_j$, where a_i and b_{ij} are constants independent of the coordinates. The number of independent components of the matrix b_{ij} is

(a) 5 (b) 6 (c) 3 (d) 4

Topic: Electromagnetic Theory

Sub Topic: Relativistic Electrodynamics (Field Tensor)

Ans. : (a)

Solution: Given that $E_i=a_i+\sum_{j=1,2,3} b_{ij}x_j$

From the given condition, the b_{ij} should be symmetric and traceless

The typical feature of b_{ij} is

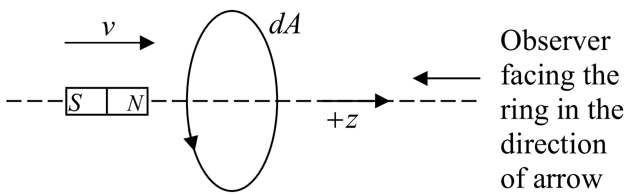
$$b_{ij} = \begin{bmatrix} u & v & w \\ v & p & s \\ w & s & r \end{bmatrix}$$

The trace less of b_{ij} means $u + p + r = 0$ (It will reduce one independent component)

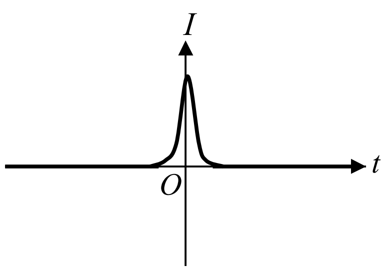
The symmetric property will reduce three independent components

Thus, b_{ij} will have 9-3-1 =5 Independent component

Q11. A conducting wire in the shape of a circle lies on the (x, y)-plane with its centre at the origin. A bar magnet moves with a constant velocity towards the wire along the z-axis (as shown in the figure below)



We take $t = 0$ to be the instant at which the midpoint of the magnet is at the centre of the wire loop and the induced current to be positive when it is counter-clockwise as viewed by the observer facing the loop and the incoming magnet. In these conventions, the best schematic representation of the induced current $I(t)$ as a function of t, is

(a)


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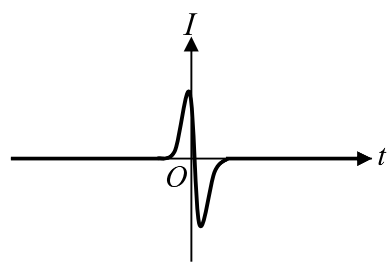
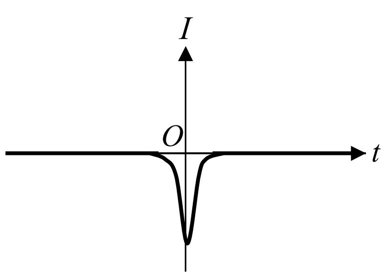
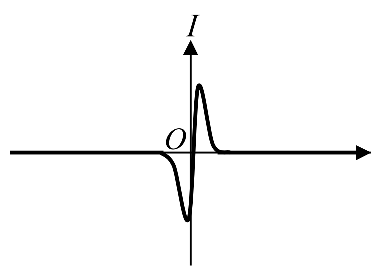


(c)


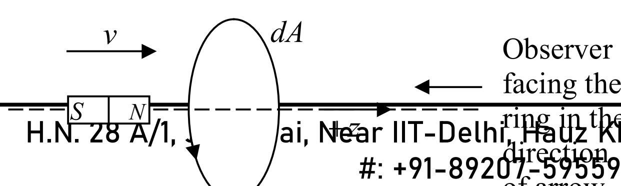
(d)


Topic: Electromagnetic Theory

Sub Topic: Magnetostatic Lenz Law

Ans. : (d)

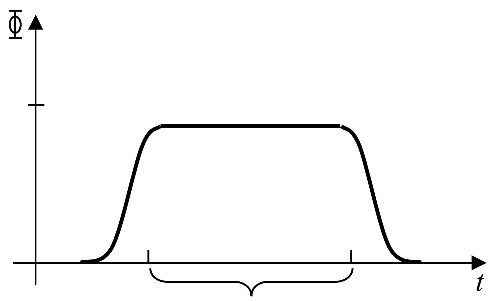
Solution:

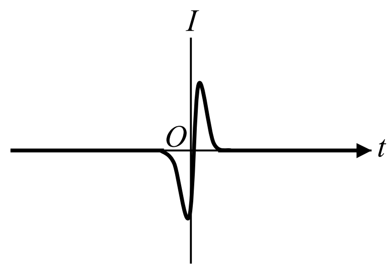


Initially, at $t=0$, the flux will be zero. Once, the bar magnet move towards circular loop, the associated flux will increase. When bar magnet reaches the circular loop, the flux attain constant. But, once bar magnet cross circular loop and move away from loop, then the associated flux will decrease.

We know that,

$$\varepsilon = -\frac{d\varphi}{dt} \text{ and } \varepsilon \,\alpha\, I$$





Q12. In an experiment to measure the charge to mass ratio $\frac{e}{m}$ of the electron by Thomson's method, the values of the deflecting electric field and the accelerating potential are $6\times10^6\ N/C$ (newton per coulomb) and $150V$, respectively. The magnitude of the magnetic field that leads to zero deflection of the electron beam is closest to

(a) $0.6\,T$ (b) $1.2\,T$ (c) $0.4\,T$ (d) $0.8\,T$

Topic: Electromagnetic Theory

Sub Topic: Magnetostat (Thomson Experiment)

Ans. : (d)

Solution: $\frac{e}{m}=\frac{1}{2}\frac{E^2}{B^2V} \Rightarrow 1.77\times10^{11}=\frac{1}{2}\times\frac{36\times10^{12}}{B^2\times150}$

$$\frac{1.77\times10^{11}\times300}{36\times10^{12}}=\frac{1}{B^2}\Rightarrow B^2=\frac{36\times10^{12}}{1.77\times3\times10^{13}}=\frac{3.6}{1.77\times3}$$

$B = 0.8\,T$

Q13. A monochromatic source emitting radiation with a certain frequency moves with a velocity v away from a stationary observer A. It is moving towards another observer B (also at rest) along a line joining the two. The frequencies of the radiation recorded by A and B are ν_A and ν_B, respectively. If the ratio $\frac{\nu_B}{\nu_A} = 7$, then the value of $\frac{v}{c}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) $\frac{\sqrt{3}}{2}$

Topic: Classical Mechanics

Sub Topic: STR Doppler effect

Ans. : (c)

Solution: Using concept of doppler effect of light. For observer B $\nu_B = \nu\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$ and for observer A

$$\nu_A = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \text{ it is given } \frac{\nu_B}{\nu_A} = \frac{\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}}{\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}} = 7 \Rightarrow \frac{1+\frac{v}{c}}{1-\frac{v}{c}} = 7 \Rightarrow \frac{v}{c} = \frac{3}{4}$$

Q14. The Hamiltonian of a particle of mass m in one-dimension is $H = \frac{1}{2m}p^2 + \lambda|x|^3$, where $\lambda > 0$ is constant. If E_1 and E_2, respectively, denote the ground state energies of the particle for $\lambda = 1$ and $\lambda = 2$ (in appropriate units) the ratio $\frac{E_2}{E_1}$ is best approximated by

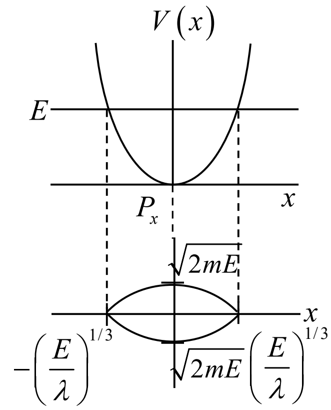
(a) 1.260 (b) 1.414 (c) 1.516 (d) 1.320

Topic: Quantum Mechanics

Sub Topic: WKB Approximation

Ans. : (d)

Solution: Quantum mechanics semiclassical method



Using Bohr Somerfield theorem $J = \oint p\,dx = nh$

$$J = 4\int_0^{\frac{E}{\lambda}} \sqrt{2m\left(E - \lambda x^3\right)}dx = h \text{ for ground state } n = 1$$

$$\sqrt{2mE}\left(\frac{E}{\lambda}\right)^{1/3} \approx h \Rightarrow E^{\frac{1}{2}+\frac{1}{3}} \propto \lambda^{\frac{1}{3}} \Rightarrow E^{\frac{5}{6}} \propto \lambda^{\frac{1}{2}} \Rightarrow E \propto \lambda^{\frac{2}{5}}$$

$$\frac{E(\lambda = 2)}{E(\lambda = 1)} = (2)^{2/5} = 1.32$$

Q15. A particle of mass m is in a one dimensional infinite potential well of length L, extending from $x = 0$ to $x = L$. When it is in the energy eigenstate labelled by $n, (n = 1, 2, 3, ...)$ the probability of finding it in the interval $0 \le x \le \frac{L}{8}$ is $\frac{1}{8}$. The minimum value of n for which this is possible is

(a) 4 (b) 2 (c) 6 (d) 8

Topic: Quantum Mechanics

Sub Topic: Particle in Box

Ans. : (a)

Solution: For particle in one dimensional box $P\left(0 \le x \le \frac{L}{8}\right) = \int_0^{\frac{L}{8}} \frac{2}{L}\sin^2\frac{n\pi x}{L}dx = \frac{1}{8}$

$$\frac{2}{L}\frac{1}{2}\int_0^{\frac{L}{8}}\left(1 - \cos\frac{2n\pi x}{L}\right)dx = \frac{1}{8} \Rightarrow \frac{1}{8} - .\frac{1}{2n\pi}\sin\frac{2n\pi}{8} = \frac{1}{8} \Rightarrow \sin\frac{n\pi}{4} = 0 \Rightarrow n = 4$$

Q16. A two-state system evolves under the action of the Hamiltonian $H = E_0|A\rangle\langle A| + (E_0 + \Delta)|B\rangle\langle B|$, where $|A\rangle$ and $|B\rangle$ are its two orthonormal states. These states transform to one another under parity i.e., $P|A\rangle = |B\rangle$ and $P|B\rangle = |A\rangle$. If at time $t = 0$

the system is in a state of definite parity $P=1$, the earliest time t at which the probability of finding the system in a state of parity $P=-1$ is one, is

(a) $\frac{\pi\hbar}{2\Delta}$ (b) $\frac{\pi\hbar}{\Delta}$ (c) $\frac{2\pi\hbar}{2\Delta}$ (d) $\frac{2\pi\hbar}{\Delta}$

Topic: Quantum Mechanics

Sub Topic: Postulates of Quantum Mechanics

Ans. : (b)

Solution: $H = E_0|A\rangle\langle A| + (E_0+\Delta)|B\rangle\langle B| \quad P|A\rangle = |B\rangle, P|B\rangle = |A\rangle$

$H|A\rangle = E_0 A, H|B\rangle = (E_0+\Delta)|B\rangle$

So $|A\rangle$ and $|B\rangle$ are eigen vector of Hamiltonian with eigen value E_0 and $E_0+\Delta$

At $t=0, |\psi\rangle = c_1|A\rangle + c_2|B\rangle$

It is given $P|\psi\rangle = \psi \Rightarrow c_1|B\rangle + c_2|A\rangle = c_1|A\rangle + c_2|B\rangle \Rightarrow c_1 = c_2 = \frac{1}{\sqrt{2}}$

Now $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle \exp\left(-\frac{iE_0 t}{\hbar}\right) + |B\rangle \exp\left(-\frac{i(E_0+\Delta)t}{\hbar}\right)\right)$

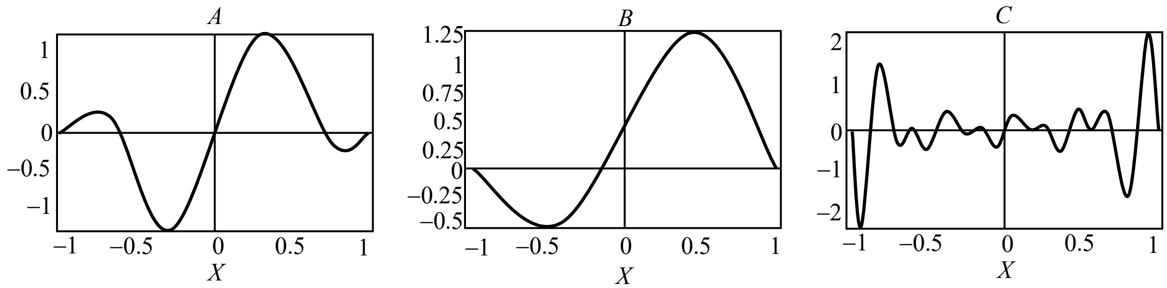
$$P|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|B\rangle \exp\left(-\frac{iE_0 t}{\hbar}\right) + |A\rangle \exp\left(-\frac{i(E_0+\Delta)t}{\hbar}\right)\right) = -|\psi(t)\rangle$$

$$\frac{1}{\sqrt{2}}\left(|B\rangle \exp\left(-\frac{iE_0 t}{\hbar}\right) + |A\rangle \exp\left(-\frac{i(E_0+\Delta)t}{\hbar}\right)\right) = -\left(\frac{1}{\sqrt{2}}\left(|A\rangle \exp\left(-\frac{iE_0 t}{\hbar}\right) + |B\rangle \exp\left(-\frac{i(E_0+\Delta)t}{\hbar}\right)\right)\right)$$

$$|B\rangle + A\exp\left(\frac{i\Delta t}{\hbar}\right) = -|A\rangle - B\exp\left(\frac{i\Delta t}{\hbar}\right) \Rightarrow \exp\left(\frac{i\Delta t}{\hbar}\right) = -1$$

$$\Rightarrow \exp\left(\frac{i\Delta t}{\hbar}\right) = -1 \Rightarrow \cos\left(\frac{\Delta t}{\hbar}\right) = -1 \Rightarrow t = \frac{\pi\hbar}{\Delta}$$

Q17. The figures below depict three different wavefunctions of a particle confined to a one-dimensional box $-1 \le x \le 1$



The wavefunctions that correspond to the maximum expectation values $|\langle x\rangle|$ (absolute value of the mean position) and $\langle x^2\rangle$, respectively, are

(a) B and C (b) B and A (c) C and B (d) A and B

Topic: Quantum Mechanics

Sub Topic: One Dimensional Box

Ans. : (a)

Solution: The probability distribution is symmetric about $x=0$ for A, C so $\langle X\rangle = 0$ for these wave function in case B the area under of probability density is more in region $x>0$ so $\langle X\rangle > 0$ in case B.

The maximum fluctuation is in C so max $\langle x^2\rangle$ in C So answer is B and C

Q18. Which of the following two physical quantities cannot be measured simultaneously with arbitrary accuracy for the motion of a quantum particle in three dimensions?

(a) square of the radial position and z-component of angular momentum $(r^2 \text{ and } L_z)$

(b) x-components of linear and angular momenta (p_x and L_x)

(c) y -component of position and z-component of angular momentum (y and L_z)

(d) squares of the magnitudes of the linear and angular momenta (p^2 and L^2)

Topic: Quantum Mechanics

Sub Topic: Angular Momentum

Ans. : (c)

Solution: $[Y, L_z] = [Y, XP_Y - YP_X] = X[Y, P_Y] = i\hbar X$

Q19. The ratio $\frac{C_P}{C_V}$ of the specific heats at constant pressure and volume of a monoatomic ideal gas in two dimensions is

(a) $\frac{3}{2}$ (b) 2 (c) $\frac{5}{3}$ (d) $\frac{5}{2}$

Topic: Thermodynamics & Statistical Mechanics

Sub Topic: Kinetic Theory of Gas

Ans. : (b)

Solution: $C_V = \frac{f}{2}R$, $f =$ D.O.F.

For monoatomic in $2D$

$$C_V = \frac{2}{2}R = R$$

We know that $C_P - C_V = R$

$$C_P = 2R$$

$$\frac{C_P}{C_V} = \frac{2R}{R} = 2$$

Q20. The volume and temperature of a spherical cavity filled with black body radiation are V and $300K$ respectively. If it expands adiabatically to a volume $2V$, its temperature will be closest to

(a) $150\ K$ (b) $300\ K$ (c) $250\ K$ (d) $240\ K$

Topic: Thermodynamics & Statistical Mechanics

Sub Topic: Black body radiation

Ans. : (d)

Solution: For adiabatic in black body

$$VT^3 = \text{Constant}$$

$$V \times (300)^3 = K \qquad \text{(i)}$$

$$2V \times (T)^3 = K \qquad \text{(ii)}$$

By dividing (ii) by (i) $T = \frac{300}{2^{1/3}} = 240$

Q21. The total number of phonon modes in a solid of volume V is $\int_0^{\omega_D} g(\omega)\, d\omega = 3N$, where N is the number of primitive cells, ω_D is the Debye frequency and density of photon modes is

$g(\omega)=AV\omega^2$ (with $A>0$ a constant). If the density of the solid doubles in a phase transition, the Debye temperature θ_D will

(a) increase by a factor of $2^{2/3}$ (b) increase by a factor of $2^{1/3}$

(c) decrease by a factor of $2^{2/3}$ (d) decrease by a factor of $2^{1/3}$

Topic: Solid State Physics

Sub Topic: Lattice Specific Heat, Lattice Vibration

Ans. : (b)

Solution: $\int_0^{\omega_D} AV\omega^2 d\omega = N \Rightarrow AV\frac{\omega_D{}^3}{3}=N \Rightarrow \omega_D{}^3 \alpha \frac{N}{V} \alpha\, \rho \Rightarrow \theta_D^3 \alpha\, \rho$ [Since, $\omega_D \alpha\, \theta_D$]

If $\rho' \to 2\rho \Rightarrow \theta'_D\ \alpha\, 2^{\frac{1}{3}}\rho^{\frac{1}{3}}$

Thus, the Debye temperature is increased by $2^{\frac{1}{3}}$

Q22. The position of a particle in one dimension changes in discrete steps. With each step it moves to the right, however, the length of the step is drawn from a uniform distribution from the interval $\left[\lambda-\frac{1}{2}w, \lambda+\frac{1}{2}w\right]$, where λ and w are positive constants. If X denotes the distance from the starting point after N steps, the standard deviation $\sqrt{\langle X^2\rangle-\langle X\rangle^2}$ for large values of N is

(a) $\frac{\lambda}{2}\times\sqrt{N}$ (b) $\frac{\lambda}{2}\times\sqrt{\frac{N}{3}}$ (c) $\frac{w}{2}\times\sqrt{N}$ (d) $\frac{w}{2}\times\sqrt{\frac{N}{3}}$

Topic: Thermodynamics & Statistical Mechanics

Sub Topic: Random Walk Problem

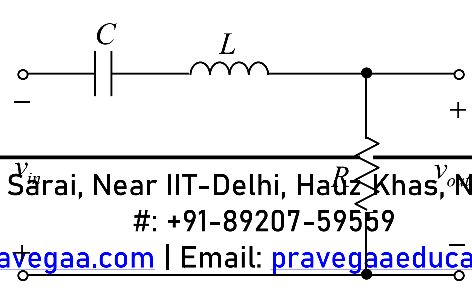
Ans. : (d)

Solution: $\sqrt{\langle X^2\rangle-\langle X\rangle^2}=\sqrt{\frac{N^2}{2}\left(\lambda-\frac{w}{2}\right)^2+\frac{N^2}{2}\left(\lambda+\frac{w}{2}\right)^2-\left[\frac{N}{2}\left(\lambda-\frac{w}{2}+\lambda+\frac{w}{2}\right)\right]^2}$

$$\sqrt{\frac{N^2}{2}\left(\lambda-\frac{w}{2}\right)^2+\frac{N^2}{2}\left(\lambda+\frac{w}{2}\right)^2-\left[\frac{N}{2}\left(\lambda-\frac{w}{2}+\lambda+\frac{w}{2}\right)\right]^2}=\sqrt{\frac{N^2w^2}{4}}=\frac{w}{2}\sqrt{N}$$

For one dimension $N=\frac{N}{3}$. Thus, $\sqrt{\langle X^2\rangle-\langle X\rangle^2}=\frac{w}{2}\sqrt{\frac{N}{3}}$

Q23. In the LCR circuit shown below, the resistance $R=0.05\ \Omega$, the inductance $L=1\ H$ and the capacitance $C=0.04\ F$.



If the input v_{in} is a square wave of angular frequency 1 rad/s the output v_{out} is best approximated by a

(a) square wave of angular frequency $1\ \text{rad/s}$

(b) sine wave of angular frequency $1\ \text{rad/s}$

(c) square wave of angular frequency $5\ \text{rad/s}$

(d) sine wave of angular frequency $5\ \text{rad/s}$

Topic: Electronics

Sub Topic: AC Circuit

Ans. : (d)

Q24. In an experiment, the velocity of a non-relativistic neutron is determined by measuring the time $(\sim 50\ ns)$ it takes to travel from the source to the detector kept at a distance L. Assume that the error in the measurement of L is negligibly small. If we want to estimate the kinetic energy T of the neutron to within 5% accuracy i.e., $\left|\frac{\delta T}{T}\right| \leq 0.05$, the maximum permissible error $|\delta T|$ in measuring the time of flight is nearest to

(a) $1.75\ ns$ (b) $0.75\ ns$ (c) $2.25\ ns$ (d) $1.25\ ns$

Topic: Electronics

Sub Topic: Experimental Technique

Ans. : (d)

Solution: $t = 50\,ns$

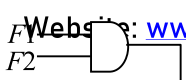
Kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{L^2}{t^2}$

Taking log, $\log T = \log\left(\frac{1}{2}m\right) + 2\log L - 2\log t$

Deviation $\frac{\Delta T}{T} = 0 + 0 + 2\frac{\Delta t}{t} \Rightarrow \frac{0.05 \times 50}{2} = \Delta t$

$\Delta t = 25 \times 0.05 = 1.25\,ns$

Q25. The door of an X - ray machine room is fitted with a sensor D (0 is open and 1 is closed). It is also equipped with three fire sensors F_1, F_2 and F_3 (each is 0 when disable and 1 when enabled).

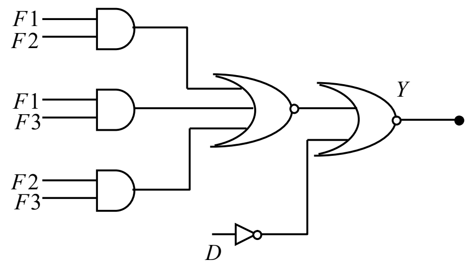


The X-ray machine can operate only if the door is closed and at least 2 fire sensors are enabled.

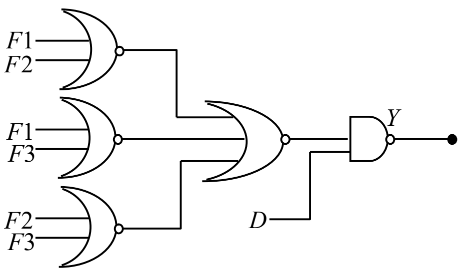
The logic circuit to ensure that the machine can be operated is

(a)

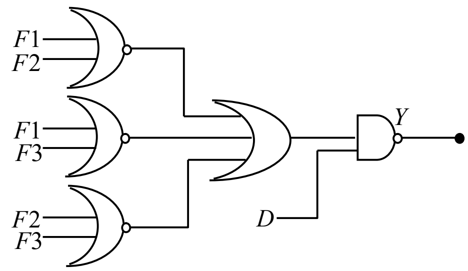
(b)



(c)



(d)



Topic: Electronics

Sub Topic: Logic Gate

Ans. : (b)

PART C

Q1. If we use the Fourier transform $\phi(x,y)=\int e^{ikx}\phi_k(y)dk$ to solve the partial differential equation

$-\frac{\partial^2\phi(x,y)}{\partial y^2}-\frac{1}{y^2}\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{m^2}{y^2}\phi(x,y)=0$ in the half-plane $\{(x,y):-\infty<x<\infty,0<y<\infty\}$ the Fourier modes $\phi_k(y)$ depend on y as y^α and y^β. The values of α and β are

(a) $\frac{1}{2}+\sqrt{1+4(k^2+m^2)}$ and $\frac{1}{2}-\sqrt{1+4(k^2+m^2)}$

(b) $1+\sqrt{1+4(k^2+m^2)}$ and $1-\sqrt{1+4(k^2+m^2)}$

(c) $\frac{1}{2}+\frac{1}{2}\sqrt{1+4(k^2+m^2)}$ and $\frac{1}{2}-\frac{1}{2}\sqrt{1+4(k^2+m^2)}$

(d) $1+\frac{1}{2}\sqrt{1+4(k^2+m^2)}$ and $1-\frac{1}{2}\sqrt{1+4(k^2+m^2)}$

Topic: Mathematical Physics

Sub Topic: Ordinary Differential Equation

Ans. : (c)

Solution: $\phi(x,y)=\int e^{ik.x}\phi_k(y)dk$

$$\phi(x,y)=\int e^{ik.x}\phi_k(y)dk \Rightarrow \frac{d^2\phi(x,y)}{dx^2}=-k^2\phi$$

Given that,

$$-\frac{d^2\phi(x,y)}{dy^2}-\frac{1}{y^2}\frac{d^2\phi(x,y)}{dx^2}+\frac{m^2}{y^2}\phi(x,y)=0$$

$$y^2\frac{d^2\phi(x,y)}{dy^2}+\frac{d^2\phi(x,y)}{dx^2}-m^2\phi(x,y)=0$$

$$y^2\frac{d^2\phi(x,y)}{dy^2}-k^2\phi(x,y)-m^2\phi(x,y)=0$$

Let, $y=e^{z}$: Cauchy-Euler Form

$$D(D-1)\phi(x,y)-k^2\phi(x,y)-m^2\phi(x,y)=0,\quad D=\frac{d}{dz}$$

$$D^2\phi(x,y)-D\phi(x,y)-k^2\phi(x,y)-m^2\phi(x,y)=0$$

$$D^2-D-(k^2+m^2)=0\Rightarrow D=\frac{1\pm\sqrt{1+4(k^2+m^2)}}{2}$$

Two roots are

$$\alpha=\frac{1+\sqrt{1+4\left(k^2+m^2\right)}}{2},\ \beta=\frac{1-\sqrt{1+4\left(k^2+m^2\right)}}{2}$$

Q2. The Newton-Raphson method is to be used to determine the reciprocal of the number $x=4$. If we start with the initial guess 0.20 then after the first iteration the reciprocal is

(a) 0.23 (b) 0.24 (c) 0.25 (d) 0.26

Topic: Mathematical Physics

Sub Topic: Numerical Techniques (Newton-Raphson Method)

Ans. : (b)

Solution: $N=4, \frac{1}{x}=N \Rightarrow \frac{1}{N}=x$

$$f(x)=\frac{1}{x}-N,\ x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\left(\frac{1}{x_n}-N\right)$$

$$\Rightarrow x_{n+1}=x_n-\frac{\left(\frac{1}{x_n}-N\right)}{\frac{-1}{x_n{}^2}} \qquad \left[\Rightarrow f'(x)=\frac{-1}{x_n{}^2}\right]$$

$$=x_n+x_n-Nx_n$$

$$x_{n+1}=x_n\left(2-Nx_n\right)=x_n+x_n\left(\frac{1}{x_n}-N\right)=x_n+x_n-Nx$$

Given $x_0=0.2$

$$\Rightarrow x_1=x_0\left[2-4\times x_0\right]=0.2\times\left[2-4\times0.2\right]=0.24$$

Q3. The Legendre polynomials $P_n(x), n=0,1,2,\ldots,$ satisfying the orthogonality condition $\int_{-1}^{1}P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{nm}$ on the interval $[-1,+1]$ may be defined by the Rodrigues formula $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$. The value of the definite integral $\int_{-1}^{1}\left(4+2x-3x^2+4x^3\right)P_3(x)\,dx$ is

(a) $\frac{3}{5}$ (b) $\frac{11}{15}$ (c) $\frac{23}{32}$ (d) $\frac{16}{35}$

Topic: Mathematical Physics

Sub Topic: Polynomials Legendre Polynomial

Ans. : (d)

Solution: $\int_{-1}^{1}\left(4+2x-3x^2+4x^3\right)P_3(x)\,dx$

$$\int_{-1}^{1}\left(4+2x-3x^2+4x^3\right)\frac{1}{2}\left(5x^3-3x\right)dx$$

Only take even function

$$\int_{-1}^{1}\left(5x^4-3x^2+10x^6-6x^4\right)dx = x^5\Big|_{-1}^{1} - x^3\Big|_{-1}^{1} + \frac{10}{7}x^7\Big|_{-1}^{1} - \frac{6}{5}x^5\Big|_{-1}^{1}$$

$$= 2-2+\frac{20}{7}-\frac{12}{5} = \frac{100-84}{35} = \frac{16}{35}$$

Q4. A particle of mass m moves in a potential that is $V = \frac{1}{2}m\left(\omega_1^2x^2+\omega_2^2y^2+\omega_3^2z^2\right)$ in the coordinates of a non-inertial frame F. The frame F is rotating with respect to an inertial frame with an angular velocity $\hat{k}\Omega$, where $\hat{k}$ is the unit vector along their common z-axis. The motion of the particle is unstable for all angular frequencies satisfying

(a) $\left(\Omega^2-\omega_1^2\right)\left(\Omega^2-\omega_2^2\right)>0$

(b) $\left(\Omega^2-\omega_1^2\right)\left(\Omega^2-\omega_2^2\right)<0$

(c) $\left(\Omega^2-\left(\omega_1+\omega_2\right)^2\right)\left(\Omega^2-\left|\omega_1-\omega_2\right|^2\right)>0$

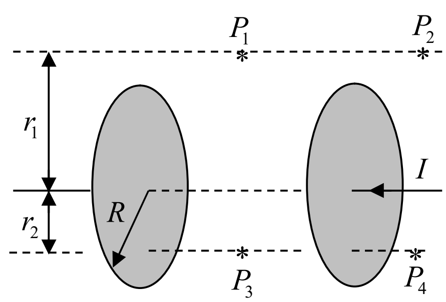
(d) $\left(\Omega^2-\left(\omega_1+\omega_2\right)^2\right)\left(\Omega^2-\left|\omega_1-\omega_2\right|^2\right)<0$

Topic: Classical Mechanics

Sub Topic: Pseudoforce

Ans. : (b)

Q5. The figure below shows an ideal capacitor consisting of two parallel circular plates of radius R. Points P_1 and P_2 are at a transverse distance $r_1 > R$ from the line joining the centres of the plates, while points P_3 and P_4 are at a transverse distance $r_2 < R$.



If $B(x)$ denotes the magnitude of the magnetic fields at these points, which of the following holds while the capacitor is charging?

(a) $B(P_1)<B(P_2)$ and $B(P_3)<B(P_4)$

(b) $B(P_1)>B(P_2)$ and $B(P_3)>B(P_4)$

(c) $B(P_1)=B(P_2)$ and $B(P_3)<B(P_4)$

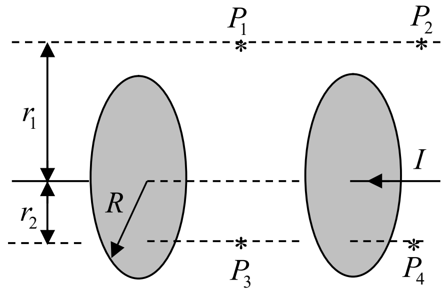
(d) $B(P_1)=B(P_2)$ and $B(P_3)>B(P_4)$

Topic: Electromagnetic Theory

Sub Topic: Maxwell Equation (Displacement Current)

Ans. : (c)

Solution:



The magnetic field at P_1 and P_3 are due to displacement current. On the other hand, the magnetic field at P_2 and P_4 are due to conduction current.

The current at P_1 and P_2 are equal i.e. $I_c = I_D$. Thus, $B(P_1) = B(P_2)$

The current at P_4 is greater than the P_3 i.e. $I_c > I_D$. Since $I_D = \int J.ds$.

Thus, $B(P_3) < B(P_4)$

Q6. A perfectly conducting fluid, of permittivity ε and permeability μ, flows with a uniform velocity v in the presence of time dependent electric and magnetic fields E and B, respectively. If there is a finite current density in the fluid, then

(a) $\nabla \times (v \times B) = \frac{\partial B}{\partial t}$

(b) $\nabla \times (v \times B) = -\frac{\partial B}{\partial t}$

(c) $\nabla \times (v \times B) = \sqrt{\varepsilon\mu}\frac{\partial E}{\partial t}$

(d) $\nabla \times (v \times B) = -\sqrt{\varepsilon\mu}\frac{\partial E}{\partial t}$

Topic: Electromagnetic Theory

Sub Topic: Maxwell's Equation

Ans. : (a)

Solution: $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) = 0 \Rightarrow (\vec{E} + \vec{v} \times \vec{B}) = \frac{\vec{J}}{\sigma} = 0$

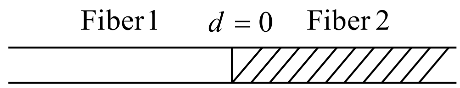
$\vec{E} = \vec{B} \times \vec{v}$

Taking curl on both sides

$$\nabla \times \vec{E} = (\nabla \times \vec{B} \times \vec{v}) \Rightarrow (\nabla \times \vec{B} \times \vec{v}) = -\frac{\partial B}{\partial t} \Rightarrow (\nabla \times \vec{v} \times \vec{B}) = \frac{\partial B}{\partial t}$$

This is known as Alfven's theorem

Q7. A laser beam propagates from fiber 1 to fiber 2 in a cavity made up of two optical fibers (as shown in the figure). The loss factor of fiber 2 is $10\, dB/km$.



If $E_2(d)$ denotes the magnitude of the electric field in fiber 2 at a distance d from the interface, the ratio $\frac{E_2(0)}{E_2(d)}$ for $d = 10\, km$, is

(a) 10^2 (b) 10^3 (c) 10^5 (d) 10^7

Topic: Electromagnetic Theory

Sub Topic: Optics

Ans. : (c)

Solution: Loss in 2nd medium = 10 dB/km

$$\log\left(\frac{P_1}{P_2}\right) = 1 \Rightarrow \frac{P_1}{P_2} = 10 \Rightarrow \left(\frac{E_1}{E_2}\right)^2 = 10 \Rightarrow \frac{E_1}{E_2} = \sqrt{10}$$

The most general formula

$$\left[\frac{E_1(d)}{E_2(d)} = \left(\sqrt{10}\right)^d\right] \Rightarrow d = 10 \Rightarrow \frac{E_0}{E_2(d)} = \left(\sqrt{10}\right)^{10} = 10^5$$

Q8. A particle in two dimensions is found to trace an orbit $r(\theta)=r_0\theta^2$. If it is moving under the influence of a central potential $V(r)=c_1r^{-a}+c_2r^{-b}$, where r_0, c_1 and c_2 are constants of appropriate dimensions, the values of a and b, respectively, are

(a) 2 and 4 (b) 2 and 3

(c) 3 and 4 (d) 1 and 3

Topic: Classical Mechanics

Sub Topic: Central Force Problem

Ans. : (b)

Solution: $r(\theta)=r_0\theta^2$

Let $u=\frac{1}{r}=\frac{1}{r_0}\theta^{-2}$

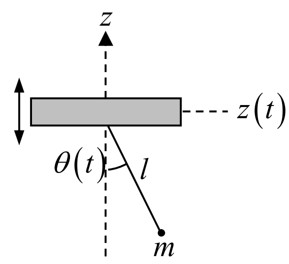
Differential equation of orbit

$$\left(\frac{d^2u}{d\theta^2}+u\right)=-\frac{m}{l^2u^2}f \Rightarrow \left[\frac{1\times 6\times\theta^{-4}}{r_0}+\frac{1}{r_0}\times\theta^{-2}\right]=-\frac{m}{l^2u^2}f$$

$$\Rightarrow\left[A_1u^2+A_2u\right]=-\frac{m}{l^2u^2}f \Rightarrow\left[B_1u^4+B_2u^3\right]=f$$

$$\left[B_1r^{-4}+B_2r^{-3}\right]=f(r)=-\frac{dV}{dr}\Rightarrow V(r)=c_1r^{-2}+c_3r^{-3}\Rightarrow a=2,\ b=3$$

Q9. The fulcrum of a simple pendulum (consisting of a particle of mass m attached to the support by a massless string of length l) oscillates vertically as $\sin z(t) = a\sin\omega t$, where ω is a constant. The pendulum moves in a vertical plane and $\theta(t)$ denotes its angular position with respect to the z-axis.



If $\ell\frac{d^2\theta}{dt^2} + \sin\theta\left(g - f(t)\right) = 0$ (where g is the acceleration due to gravity) describes the equation of motion of the mass, then $f(t)$ is

(a) $a\omega^2\cos\omega t$ (b) $a\omega^2\sin\omega t$

(c) $-a\omega^2\cos\omega t$ (d) $-a\omega^2\sin\omega t$

Topic: Classical Mechanics

Sub Topic: Lagrangian of a System

Ans. : (d)

Solution: The generalised coordinate is $(l\sin\theta, -l\cos\theta + z)$

Lagrangian is given by $L = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{z}^2 - l\dot{\theta}\dot{z}\sin\theta\right) + mgl\cos\theta - z$

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} - ml\dot{z}\sin\theta, \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta} - ml\ddot{z}\sin\theta - ml\dot{z}\cos\theta\dot{\theta}$$

$$\frac{\partial L}{\partial\theta} = -ml\dot{\theta}\dot{z}\cos\theta - mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow ml^2\ddot{\theta} - ml\ddot{z}\sin\theta - ml\dot{z}\dot{\theta}\cos\theta + ml\dot{z}\dot{\theta}\cos\theta + mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} - ml\ddot{z}\sin\theta + mgl\sin\theta = 0 \Rightarrow l\ddot{\theta} - \ddot{z}\sin\theta + g\sin\theta = 0$$

Put $z = a\sin\omega t \Rightarrow \ddot{z} = -a\omega^2\sin\omega t$

$$l\ddot{\theta} + a\omega^2\sin\omega t\sin\theta + g\sin\theta = 0 \Rightarrow l\ddot{\theta} + \sin\theta\left(a\omega^2\sin\omega t + g\right) = 0$$

In problem it is given by

$$l\ddot{\theta} + \sin\theta\left(g - f(t)\right) \text{ so } f(t) = -a\omega^2\sin\omega t$$

Q10. A satellite of mass m orbits around earth in an elliptic trajectory of semi-major axis α. At a radial distance $r = r_0$, measured from the centre of the earth, the kinetic energy is equal to half the magnitude of the total energy. If M denotes the mass of the earth and the total energy is $-\frac{GMm}{2a}$, the value of $\frac{r_0}{a}$ is nearest to

(a) 1.33 (b) 1.48 (c) 1.25 (d) 1.67

Topic: Classical Mechanics

Sub Topic: Central Force Problem

Ans. : (a)

Solution: $E = T + V \Rightarrow -\frac{GMm}{2a} = \frac{GMm}{4a} - \frac{GMm}{r_0} \Rightarrow -\frac{1}{2a} - \frac{1}{4a} = -\frac{1}{r_0}$

$$-\frac{3}{4a} = -\frac{1}{r_0} \Rightarrow \frac{r_0}{a} = \frac{4}{3} = 1.33$$

Q11. A particle of mass m in one dimension is in the ground state of a simple harmonic oscillator described by a Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ in the standard notation. An impulsive force at time $t = 0$ suddenly imparts a momentum $p_0 = \sqrt{\hbar m\omega}$ to it. The probability that the particle remains in the original ground state is

(a) e^{-2} (b) $e^{-3/2}$ (c) e^{-1} (d) $e^{-1/2}$

Topic: Quantum Mechanics

Sub Topic: Harmonic Oscillator

Ans. : (d)

Solution: The ground state wave function of Harmonic oscillator

$$|\psi_{gs}\rangle = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{\alpha^2 x^2}{2}}$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{P_0^2}{2m}$$

New wave function is due combination of harmonic oscillator and free particle to impulsive force Since, the wave function is multiplicative in nature. Therefore, we can write

$$|\psi_{new}\rangle = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{\alpha^2 x^2}{2}} e^{\frac{ip_0 x}{\hbar}}$$

Probability of finding in the original ground state is

$$\langle\psi_{new}||\psi_{gs}\rangle = |\langle\psi_{new}||\psi_{gs}\rangle|^2$$

Now

$$\langle\psi_{new}||\psi_{gs}\rangle=\int_{-\infty}^{\infty}\left(\frac{\alpha}{\pi}\right)^{1/2}e^{-\frac{\alpha^2x^2}{2}}e^{\frac{-ip_0x}{\hbar}}\left(\frac{\alpha}{\pi}\right)^{1/2}e^{-\frac{\alpha^2x^2}{2}}dx=\left(\frac{\alpha}{\sqrt{\pi}}\right)\int_{-\infty}^{\infty}e^{-(\alpha^2x^2+\frac{ip_0x}{\hbar})}dx$$

$$=\left(\frac{\alpha}{\sqrt{\pi}}\right)e^{-\frac{(\frac{p_0}{\hbar})^2}{4\alpha^2}}\left(\frac{\sqrt{\pi}}{\alpha}\right)\qquad\left[Sine,\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(\alpha x^2+\beta x)}dx=e^{\frac{\beta^2}{4\alpha}}\sqrt{\frac{\pi}{\alpha}}\right]$$

Now Substitute the value of $p_0^{\,2}=m\omega\hbar,\ \alpha^2=\frac{m\omega}{\hbar}$

$\langle\psi_{new}||\psi_{gs}\rangle=e^{-\frac{m\omega}{4m\omega}}=e^{-\frac{1}{4}}$ Thus, the probability $\langle\psi_{new}||\psi_{gs}\rangle^2=e^{-\frac{1}{2}}$

Q12. The energies of a two-state quantum system are E_0 and $E_0+\alpha\hbar$, (where $\alpha>0$ is a constant) and the corresponding normalized state vectors are $|0\rangle$ and $|1\rangle$, respectively. At time $t=0$, when the system is in the state $|0\rangle$, the potential is altered by a time independent term V such that $\langle 1|V|0\rangle=\frac{\hbar\alpha}{10}$. The transition probability to the state $|1\rangle$ at times $t<<\frac{1}{\alpha}$, is

(a) $\frac{\alpha^2t^2}{25}$ (b) $\frac{\alpha^2t^2}{50}$ (c) $\frac{\alpha^2t^2}{100}$ (d) $\frac{\alpha^2t^2}{200}$

Topic: Quantum Mechanics

Sub Topic: Time Dependent Perturbation Theory

Ans. : (c)

Solution: $\langle 0|V|1\rangle=\frac{\alpha\hbar}{100}$ $P(0\to1)=\frac{4|\langle 0|V|1\rangle|^2}{\hbar^2\omega_0^2}\sin^2\frac{\omega_0t}{2}$

for small time $t\to0$

$$\frac{4|\langle 0|V|1\rangle|^2}{\hbar^2\omega_0^2}.\frac{\omega_0^2t^2}{4}=\frac{|\langle 0|V|1\rangle|^2}{\hbar^2}.t^2=\frac{\alpha^2t^2\hbar^2}{100\hbar^2}=\frac{\alpha^2t^2}{100}$$

Q13. In an elastic scattering process at an energy E, the phase shift $\delta_0\approx30^0,\delta_1\approx10^0$, while the other phase shifts are zero. The polar angle at which the differential cross section peaks is closest to

(a) 20^0 (b) 10^0 (c) 0^0 (d) 30^0

Topic: Quantum Mechanics

Sub Topic: Scattering

Ans. : (c)

Q14. The unnormalized wavefunction of a particle in one dimension in an infinite square well with walls at $x=0$ and $x=a$, is $\psi(x)=x(a-x)$. If $\psi(x)$ is expanded as a linear combination of the energy eigenfunctions, $\int_0^a |\psi(x)|^2 dx$ is proportional to the infinite series

(you may use $\int_0^a t\sin t\,dt=-a\cos a+\sin a$ and $\int_0^a t^2\sin t\,dt=-2-(a^2-2)(\cos a+2a\sin a)$)

(a) $\sum_{n=1}^{\infty}(2n-1)^{-6}$ (b) $\sum_{n=1}^{\infty}(2n-1)^{-4}$

(c) $\sum_{n=1}^{\infty}(2n-1)^{-2}$ (d) $\sum_{n=1}^{\infty}(2n-1)^{-8}$

Topic: Quantum Mechanics

Sub Topic: Particle in Box

Ans. : (a)

Solution: The normalize wavefunction can be written as

$$\psi = Ax(a-x)$$

Where A is normalization constant.

$$\int_0^a \psi^2 dx=1,\ \int_0^a [Ax(a-x)]^2 dx=1 \quad \Rightarrow A=\sqrt{\frac{30}{a^5}}$$

This wavefunction can be written in basis form

$$\psi=\sum_n c_n\phi_n,$$

The nth coefficient for infinite square well can be written as follows

$$c_n=\sqrt{\frac{2}{a}}\int_0^a \sin(\frac{n\pi x}{a})\sqrt{\frac{30}{a^5}}x(a-x)dx=\frac{2\sqrt{15}}{a^3}\left[a\int_0^a x\sin(\frac{n\pi x}{a})dx-\int_0^a x^2\sin(\frac{n\pi x}{a})dx\right]$$

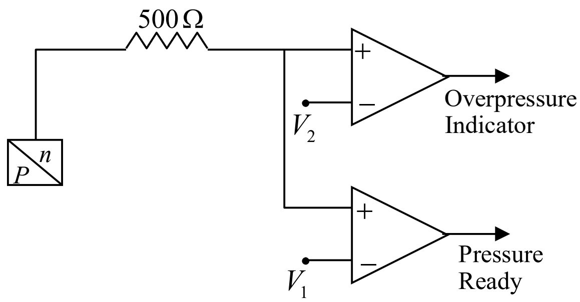
$$=\frac{2\sqrt{15}}{a^3}\left\{a\left[\left(\frac{a}{n\pi}\right)^2\sin(\frac{n\pi x}{a})-\frac{ax}{n\pi}\cos(\frac{n\pi x}{a})\right]\Bigg|_0^a-\left[\left\{2\left(\frac{a}{n\pi}\right)^2 x\sin(\frac{n\pi x}{a})-\frac{(\frac{n\pi x}{a})^2-2}{(\frac{n\pi}{a})^3}\cos(\frac{n\pi x}{a})\right\}\right]_0^a\right.$$

$$c_n=\frac{4\sqrt{15}}{(n\pi)^3}[\cos(0)-\cos(n\pi)]\Rightarrow c_n=\begin{cases}0, & for\ even\,n\\ \frac{8\sqrt{15}}{(n\pi)^3}, & for\ odd\ n\end{cases}$$

Thus, $c_n \alpha \sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}$

Now, $\int|\psi|^2 dx\,\alpha |c_n|^2\ \alpha \sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}$

Q15. The pressure of a gas in a vessel needs be maintained between 1.5 bar to 2.5 bar in an experiment. The vessel is fitted with a pressure transducer that generates $4\,mA$ to $20\,mA$ current for pressure in the range 1 bar to 5 bar. The current output of the transducer has a linear dependence on the pressure.



The reference voltages V_1 and V_2 in the comparators in the circuit (shown in figure above) suitable for the desired operating conditions, are, respectively

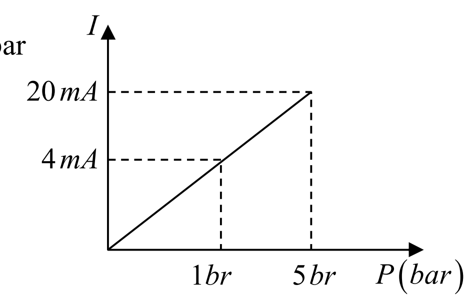
(a) $2\,V$ and $10\,V$ (b) $2\,V$ and $5\,V$ (c) $3\,V$ and $10\,V$ (d) $3\,V$ and $5\,V$

Topic: Electronics

Sub Topic: OP-AMP

Ans. : (d)

Solution: Slope $=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{20-4}{5-1}=\dfrac{16}{4}=4$ mA/bar



$1.5\,bar = 4\times 2.5 = 6\,mA$

$2.5\,bar = 4\times 2.5 = 10\,mA$

$V_1 = 6mA\times 500 = 3V$

$V_2 = 10mA\times 500 = 5V$

Q16. The nuclei of ^{137}Cs decay by the emission of β -particles with a half of 30.08 years. The activity (in units of disintegrations per second or Bq) of a 1 mg source of ^{137}Cs, prepared on January 1, 1980 as measured on January 1, 2021 is closest to

(a) 1.79×10^{16} (b) 1.79×10^{9} (c) 1.24×10^{16} (d) 1.24×10^{9}

Topic: Nuclear & Particle Physics

Sub Topic: Radioactive Decay

Ans. : (d)

Solution: 137 gm Cs contains 6.023×10^{23} atoms

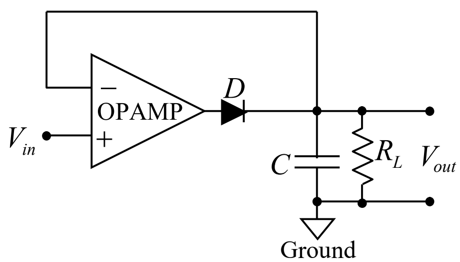
1 gm Cs contains $\dfrac{6.023\times 10^{23}}{137}$ atoms

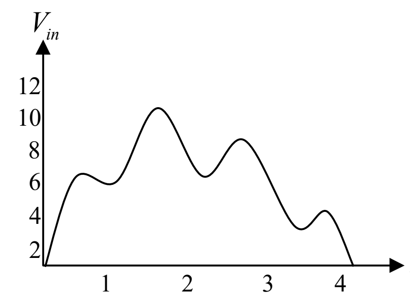
1 mg Cs contains $\dfrac{6.023\times10^{23}\times10^{-3}}{137}=N_0$ atoms

Activity: $N_0\lambda e^{-\lambda t}=\dfrac{6.023\times0.693\times10^{23}\times10^{-3}}{137}e^{-\frac{0.693\times41}{30.08}}$

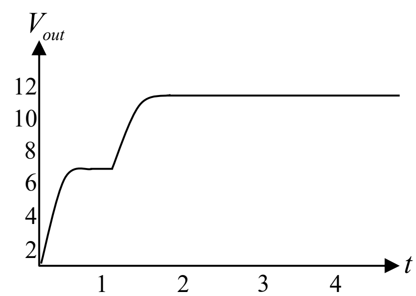
$$=\frac{6.023\times0.693\times10^{20}\times e^{-\frac{0.693\times41}{30.08}}}{137\times30.08\times365\times24\times3600}=\frac{6.023\times\times0.693\times10^{20-9}\times e^{-\frac{0.693\times41}{30.08}}}{1.37\times3.008\times3.65\times2.4\times3.6}=1.24\times10^{9}$$

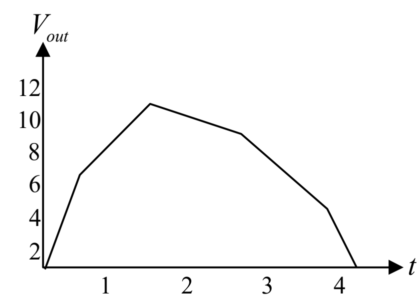
Q17. In the following circuit the input voltage V_{in} is such that $|V_{in}|<|V_{out}|$, where V_{sat} is the saturation voltage of the op-amp. (Assume that the diode is an ideal one and R_LC is much large than the duration of the measurement.)

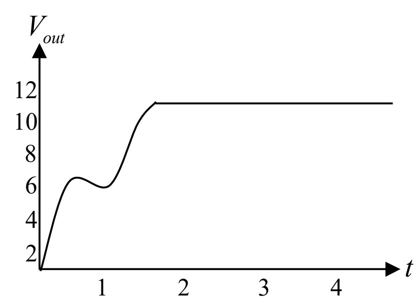


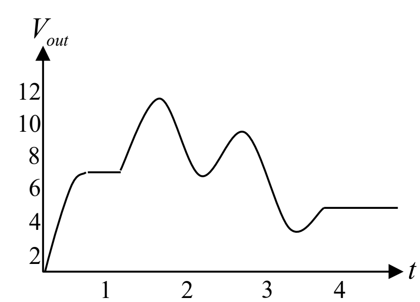


for the input voltage as shown in the figure above, the output voltage V_{out} is best represented by

(a)


(b)


(c)


(d)


for the input voltage as shown in the figure above, the output voltage V_{out} is best represented by

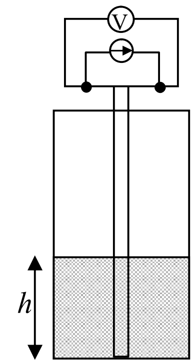
Topic: Electronics

Sub Topic: OPAMP

Ans. : (a)

Solution: Given that R_{LC} is much larger than the duration of measurement. Once capacitor is charged to input value $(v_i = 6V)$, It will be remain at $6\,V$ until further increase in voltage to $11V$ which make output to remain fixed at $11V$.

Q18. To measure the height h of a column of liquid helium in a container, a constant current I is sent through an $NbTi$ wire of length l, as shown in the figure. The normal state resistance of the $NbTi$ wire is R. If the superconducting transition temperature of $NbTi$ is $\approx 10\ K$ then the measured voltage $V(h)$ is best described by the expression



(a) $IR\left(\frac{1}{2}-\frac{2h}{l}\right)$ (b) $IR\left(1-\frac{h}{l}\right)$

(c) $IR\left(\frac{1}{2}-\frac{h}{l}\right)$ (d) $IR\left(1-\frac{2h}{l}\right)$

Topic: Solid State Physics

Sub Topic: Superconductivity

Ans. : (d)

Solution: From given diagram it is clearly noticeable that if liquid Helium is fully occupied then the maximum height will be $h_{max} = l/2$. At this time, voltage $V(h)$ will be zero due to the superconducting state.

From given option, only (a) will satisfy that $V(h_{max}) = IR\left(1-\frac{2l/2}{l}\right) = 0$.

Also, when $h = 0$ it will completely behave like a normal conductor. Which will satisfy the Ohm's Law i.e., $V = IR$.

$$V(h) = IR(1-\frac{2h}{l}) \Rightarrow V(h) = IR(1-\frac{2\times 0}{l}) = IR$$

Q19. The energy levels of a non-degenerate quantum system are $\in_n = nE_0$, where E_0 is a constant and $n = 1, 2, 3, \ldots$. At a temperature T, the free energy F can be expressed in terms of the average energy E by

(a) $E_0 + k_B T \ln\frac{E}{E_0}$ (b) $E_0 + 2k_B T \ln\frac{E}{E_0}$

(c) $E_0 - k_B T \ln \frac{E}{E_0}$ (d) $E_0 - 2k_B T \ln \frac{E}{E_0}$

Topic: Thermodynamics & Statistical Mechanics

Sub Topic: Concept of Free Energy in Statistical Physics

Ans. : (c)

Solution: Partition function can be written as follows

$$Z = e^{-\beta E_0} + e^{-2\beta E_0} + e^{-3\beta E_0} + ... = e^{-\beta E_0}(1 + e^{-\beta E_0} + e^{-2\beta E_0}..............)$$

$$= \frac{e^{-\beta E_0}}{\left(1 - e^{-\beta E_0}\right)}$$

Average energy can be written as follows

$$E = -\frac{d}{d\beta}\ln(Z) = -\frac{d}{d\beta}\ln\left(\frac{e^{-\beta E_0}}{\left(1 - e^{-\beta E_0}\right)}\right)$$

$$E = -\frac{d}{d\beta}\ln\left(\frac{e^{-\beta E_0}}{\left(1 - e^{-\beta E_0}\right)}\right) = -\frac{d}{d\beta}\left[-\beta E_0 - \ln\left(1 - e^{-\beta E_0}\right)\right] = E_0 + \frac{E_0 e^{-\beta E_0}}{\left(1 - e^{-\beta E_0}\right)} = \frac{E_0}{\left(1 - e^{-\beta E_0}\right)}$$

$$\frac{E}{E_0} = \frac{1}{\left(1 - e^{-\beta E_0}\right)} \Rightarrow \ln\left(\frac{E}{E_0}\right) = -\ln\left(1 - e^{-\beta E_0}\right)$$

Free energy can be written as follows

$$F = -kT\ln(Z) = -kT\ln\frac{e^{-\beta E_0}}{\left(1 - e^{-\beta E_0}\right)} = -kT\ln e^{-\beta E_0} + kT\ln\left(1 - e^{-\beta E_0}\right)$$

$$F = E_0 + kT\ln\left(1 - e^{-\beta E_0}\right) = E_0 - kT\ln\left(\frac{E}{E_0}\right) \text{ [Since } \ln\left(1 - e^{-\beta E_0}\right) = -\ln\left(\frac{E}{E_0}\right)\text{]}$$

Q20. A polymer made up of N monomers, is in thermal equilibrium at temperature T. Each monomer could be of length a or $2a$. The first contributes zero energy, while the second one contributes $\in$. The average length (in units of Na) of the polymer at temperature $T = \frac{\in}{k_B}$ is

(a) $\frac{5+e}{4+e}$ (b) $\frac{4+e}{3+e}$ (c) $\frac{3+e}{2+e}$ (d) $\frac{2+e}{1+e}$

Topic: Thermodynamics & Statistical Mechanics

Sub Topic: Boltzmann's Distribution Law

Ans. : (d)

Solution: Probability of length a is P_a

Probability of length 2a is P_{2a}

We know that the probability $P \alpha e^{-\beta\varepsilon}$

Thus, $\frac{P_{2a}}{P_a} = e^{-\beta\varepsilon} = e^{-\frac{1}{kT}\varepsilon}$

Given that, $T = \frac{\varepsilon}{k}$

$P_{2a} = e^{-1}P_a$......(1)

Also, total probability

$P_{2a} + P_a = 1$.......(2)

From (1) and (2), we will get

$e^{-1}P_a + P_a = 1 \Rightarrow P_a = \frac{e}{e+1}$. Thus, $P_{2a} = \frac{1}{e+1}$ [Since $p_{2a} = ep_a$]

Average length,

$$\langle l \rangle = \sum_i l_i P_i = NaP_a + 2NaP_{2a} = Na\left(P_a + 2P_{2a}\right) = Na\left(\frac{e}{e+1} + 2\frac{1}{e+1}\right) = Na\left(\frac{e+2}{e+1}\right)$$

In the unit of Na it will be $\left(\frac{e+2}{e+1}\right)$

Q21. Balls of ten different colours labeled by $a = 1, 2, .., 10$ are to be distributed among different coloured boxes. A ball can only go in a box of the same colour, and each box can contain at most one ball. Let n_a and N_a denote, respectively the numbers of balls and boxes of colour a. Assuming that $N_a >> n_a >> 1$, the total entropy (in units of the Boltzmann constant) can be best approximated by

(a) $\sum_a \left(N_a \ln N_a + n_a \ln n_a - \left(N_a - n_a\right)\ln\left(N_a - n_a\right)\right)$

(b) $\sum_a \left(N_a \ln N_a - n_a \ln n_a - \left(N_a - n_a\right)\ln\left(N_a - n_a\right)\right)$

(c) $\sum_a \left(N_a \ln N_a - n_a \ln n_a + \left(N_a - n_a\right)\ln\left(N_a - n_a\right)\right)$

(d) $\sum_a \left(N_a \ln N_a + n_a \ln n_a + \left(N_a - n_a\right)\ln\left(N_a - n_a\right)\right)$

Topic: Thermodynamics & Statistical Mechanics

Sub Topic: Microcanonical Ensemble

Ans. : (b)

Solution: Number of ways such that n_a particle will be adjusted in same colour N_a box

$$W = \prod_a \frac{\lfloor N_a}{\lfloor n_a \lfloor N_a - n_a}$$

So, entropy $S = k_B \ln W = k_B \left[\ln \lfloor N_a - \ln \lfloor n_a - \ln \lfloor N_a - n_a \right]$

Using Sterling formula $\ln \lfloor n = n \ln - n$

$$S = k_B \left[N_a \ln N_a - N_a - n_a \ln n_a + n_a - (N_a - n_a) \ln (N_a - n_a) + (N_a - n_a) \right]$$

$$S = k_B \left(N_a \ln N_a - n_a \ln n_a - (N_a - n_a) \ln (N_a - n_a) \right)$$

Total entropy $S = k_B \sum \left(N_a \ln N_a - n_a \ln n_a - (N_a - n_a) \ln (N_a - n_a) \right)$

Total entropy in the unit of k_B

$$S = \sum \left(N_a \ln N_a - n_a \ln n_a - (N_a - n_a) \ln (N_a - n_a) \right)$$

Q22. The dispersion relation of a gas of non-interacting bosons in d dimensions is $E(k) = ak^s$, where a and s are positive constants. Bose-Einstein condensation will occur for all values of

(a) $d > s$ (b) $d + 2 > s > d - 2$

(c) $s > 2$ independent of d (d) $d > 2$ independent of s

Topic: Thermodynamics & Statistical Mechanics

Sub Topic: Bose-Einstein Condensation

Ans. : (a)

Solution: Given that $E(k) = Ak^s$

Now, the density of states $\rho(E) \alpha E^{d/s-1}$, $d =$ dimension

For Bose Einstein condensation

$d/s - 1 > 0 \Rightarrow d/s > 1 \Rightarrow d > s$

Q23. Lead is superconducting below $7\,K$ and has a critical magnetic field 800×10^{-4} tesla close to $0\,K$. At $2\,K$ the critical current that flows through a long lead wire of radius $5\,mm$ is closest to

(a) $1760\,A$ (b) $1670\,A$ (c) $1950\,A$ (d) $1840\,A$

Topic: Solid State Physics

Sub Topic: Superconductivity

Ans. : (d)

Solution: $T_C = 7K$

$H_C(0) = 800\times10^{-4} = 8\times10^{-2}\times79700\, A/m$

$H_C(T=2K) = 8\times10^{-2}\left\{1-\left(\frac{2}{7}\right)^2\right\}\times79700$

$= 585551.02\times10^{-2}\, A/m$

$I_C = 2\pi r H_C = 2\times3.14\times5\times10^{-3}\times585551.02\times10^{-2}\, A = 1840\, A$

Q24. Potassium chloride forms an FCC lattice, in which K and Cl occupy alternating sites. The density of KCl is $1.98\ g/cm^3$ and the atomic weights of K and Cl are 39.1 and 35.5, respectively. The angles of incidence (in degrees) for which Bragg peaks will appear when X-ray of wavelength $0.4\, nm$ is shone on a KCl crystal are

(a) $18.5, 39.4$ and 72.2 (b) 19.5 and 41.9

(c) $12.5, 25.7, 40.5$ and 60.0 (d) $13.5, 27.8, 44.5$ and 69.0

Topic: Solid State Physics

Sub Topic: Crystal Structure, X-Ray Diffraction

Ans. : (a)

Solution: We know for cubic lattice, $a^3 = \frac{n_{eff1}M_{A1}+n_{eff2}M_{A2}}{\rho N_A} = \frac{4(39.1)+4(35.5)}{1.98\times6.23\times10^{23}} = 242\times10^{-24}\, cm$

$a = 6.22 A^0$

The interplanar spacing can be written as follows

$d = \frac{a}{\sqrt{h^2+k^2+l^2}}$

For KCl crystal, the (h, k, l) values for which Bragg's diffraction observed are (2, 0, 0), (2, 2, 0), (2, 2, 2),

From Bragg's law, $2d\sin(\theta) = \lambda$

Given that, $\lambda = 0.4 nm = 4 A^0$

$2\frac{a}{\sqrt{h^2+k^2+l^2}}\sin(\theta) = \lambda \Rightarrow 2\frac{6.22}{\sqrt{h^2+k^2+l^2}}\sin(\theta) = 4$

$\Rightarrow \frac{1}{\sqrt{h^2+k^2+l^2}}\sin(\theta) = \frac{4}{6.22\times2} = 0.31$

$\Rightarrow \sin(\theta) = 0.31\sqrt{h^2+k^2+l^2} \Rightarrow \theta = \sin^{-1}\left(0.31\sqrt{h^2+k^2+l^2}\right)$

For (h, k, l) =(2, 0, 0),

$\Rightarrow \theta = \sin^{-1}(0.31\times2) = \sin^{-1}(0.62) = 39.4^0$

Q25. In the reaction $p+n \to p+K^{+}+X$, mediated by strong interaction, the baryon number B, strangeness S and third component of isospin I_3 of the particle X are, respectively

(a) $-1,-1$ and -1 (b) $+1,-1$ and -1

(c) $+1,-2$ and $-\frac{1}{2}$ (d) $-1,-1$ and 0

Topic: Nuclear & Particle Physics

Sub Topic: Fundamental Interaction in Particle Physics

Ans. : (b)

Solution: $p+n \to p+K^{+}+X$

$B:\ 1+1=1+0+B \qquad \Rightarrow B=1$

$S:\ 0+0=0+1+S \qquad \Rightarrow S=-1$

$I_3:\ (\frac{1}{2})+(-\frac{1}{2})=(\frac{1}{2})+(\frac{1}{2})+I_3 \Rightarrow I_3=-1$

Q26. A ^{60}Co nucleus β-decays from its ground state with $J^P=5^+$ to a state of ^{60}Ni with $J^P=4^+$. From the angular momentum selection rules, the allowed values of the orbital angular momentum L and the total spin S of the election-antineutrino pair are

(a) $L=0$ and $S=1$ (b) $L=1$ and $S=0$

(c) $L=0$ and $S=0$ (d) $L=1$ and $S=1$

Topic: Nuclear & Particle Physics

Sub Topic: Beta Decay

Ans. : (a)

Solution: $^{60}Co \to {}^{60}Ni+e^{-}+\overline{v}_e$

$5^{+} \to 4^{+}+(e^{-}+\overline{v}_e)$

We know that the parity is defined as $(-1)^L$

From parity conservation, we can say that the orbital angular momentum of $(e^{-}+\overline{v}_e)$ is zero.

On the, the other hand e^{-} and $\overline{v}_e$ both are spin ½ (**Fermions**)

$^{60}Co \to {}^{60}Ni+e^{-}+\overline{v}_e$

$5^{+} \to 4^{+}+(e^{-}+\overline{v}_e)$

Total S of $(e^{-}+\overline{v}_e)$ is either 0 or 1

But, from angular momentum conservation S should (+1)

So total $S = 1/2 + 1/2 = 1$ $\quad L = 0,\ S = 1$

Q27. The Q -value of the α -decay of ^{232}Th to the ground state of ^{228}Ra is $4082\ keV$. The maximum possible kinetic energy of the α -particle is closest to

(a) $4082\,keV$ (b) $4050\,keV$ (c) $4035\,keV$ (d) $4012\,keV$

Topic: Nuclear & Particle Physics

Sub Topic: Alpha Decay

Ans. : (d)

Solution: $^{232(A)}Th \rightarrow {}^{228(A-4)}Ra + {}^{4}He$ (alpha)

$$E_{th} = \frac{(A-4)}{A}Q = \frac{228}{232}4082 = 4012\,\text{keV}$$

Q28. The $|3,0,0\rangle$ state (in the standard notation $|n,l,m\rangle$) of the H-atom in the non-relativistic theory decays to the state $|1,0,0\rangle$ via two dipole transitions. The transition route and the corresponding probability are

(a) $|3,0,0\rangle \rightarrow |2,1,-1\rangle \rightarrow |1,0,0\rangle$ and $\frac{1}{4}$ (b) $|3,0,0\rangle \rightarrow |2,1,1\rangle \rightarrow |1,0,0\rangle$ and $\frac{1}{4}$

(c) $|3,0,0\rangle \rightarrow |2,1,0\rangle \rightarrow |1,0,0\rangle$ and $\frac{1}{3}$ (d) $|3,0,0\rangle \rightarrow |2,1,0\rangle \rightarrow |1,0,0\rangle$ and $\frac{2}{3}$

Topic: Atomic & Molecular Physics

Sub Topic: Dipole Transition

Ans. : (c)

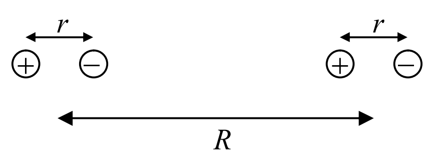
Solution: Selection rule $\Delta l = \pm 1,\ \Delta m_l = 0, \pm 1$

The possible decay mode for $|3,0,0\rangle$

$|3,0,0\rangle \rightarrow |2,1,1\rangle$, $|3,0,0\rangle \rightarrow |2,1,0\rangle$, $|3,0,0\rangle \rightarrow |2,1,-1\rangle$

Each are equally probable. So, probability for $|3,0,0\rangle \rightarrow |2,1,0\rangle$ will be 1/3

Q29. A linear diatomic molecule consists of two identical small electric dipoles with an equilibrium separation R , which is assumed to be a constant. Each dipole has charges $\pm q$ of mass m separated by r when the molecule is at equilibrium. Each dipole can execute simple harmonic motion of angular frequency ω.



Recall that the interaction potential between two dipoles of moments p_1 and p_2, separated by $R_{12} = R_{12}\hat{n}$ is $\left(p_1 \cdot p_2 - 3\left(p_1 \cdot \hat{n}\right)\left(p_2 \cdot \hat{n}\right)\right)/\left(4\pi \in_0 R_{12}^3\right)$.

Assume that $R >> r$ and let $\Omega^2 = \dfrac{q^2}{4\pi \in_0 mR^3}$. The angular frequencies of small oscillations of the diatomic molecule are

(a) $\sqrt{\omega^2 + \Omega^2}$ and $\sqrt{\omega^2 - \Omega^2}$ (b) $\sqrt{\omega^2 + 3\Omega^2}$ and $\sqrt{\omega^2 - 3\Omega^2}$

(c) $\sqrt{\omega^2 + 4\Omega^2}$ and $\sqrt{\omega^2 - 4\Omega^2}$ (d) $\sqrt{\omega^2 + 2\Omega^2}$ and $\sqrt{\omega^2 - 2\Omega^2}$

Topic: Electromagnetic Theory

Sub Topic: Dipole Interaction

Ans. : (c)

Solution: The interaction energy between electric dipole is given as

$$U = \frac{\left(p_1.p_2 - 3\left(p_1.\hat{n}\right)\left(p_2.\hat{n}\right)\right)}{4\pi\varepsilon_0 R^3{}_{12}}$$

Since, $p_1 = qr, p_2 = qr, p_1 \parallel p_2$

$$U = \frac{\left(p_1.p_2 - 3\left(p_1.\hat{n}\right)\left(p_2.\hat{n}\right)\right)}{4\pi\varepsilon_0 R^3{}_{12}} = \frac{1}{4\pi\varepsilon_0 R^3{}_{12}}(1-3)q^2r^2 = -2m\Omega^2 r^2$$

The potential energy of one dipole will be

$$U = \frac{1}{2}m\omega^2 r^2 + 2m\Omega^2 r^2 = \frac{1}{2}m\omega'^2 r^2 \Rightarrow \omega' = \sqrt{\omega^2 + 4\Omega^2}$$

The potential energy of another dipole should be

$$U = \frac{1}{2}m\omega^2 r^2 - 2m\Omega^2 r^2 = \frac{1}{2}m\omega'^2 r^2 \Rightarrow \omega' = \sqrt{\omega^2 - 4\Omega^2} \qquad \text{[From energy conservation]}$$

Q30. Diffuse hydrogen gas within a galaxy may be assumed to follow a Maxwell distribution at temperature $10^6\,K$, while the temperature appropriate for the H gas in the inter-galactic space, following the same distribution, may be taken to be $10^4\,K$. The ratio of thermal broadening $\Delta v_G / \Delta v_{1G}$ of the Lyman-α line from the H-atoms within the galaxy to that from the inter-galactic space is closest to

(a) 100 (b) $\frac{1}{100}$ (c) 10 (d) $\frac{1}{10}$

Topic: Atomic & Molecular Physics

Sub Topic: Broadening of Spectral Lines

Ans. (c)

Solution: The thermal means Doppler broadening

We know that

$$\Lambda v_D = \frac{v_0}{c}\sqrt{\frac{2kT}{m}} \Rightarrow \Lambda v_D \alpha \sqrt{T}$$

$$\frac{\Lambda v_G}{\Lambda v_{IG}} = \frac{\sqrt{10^6}}{\sqrt{10^4}} = 10$$