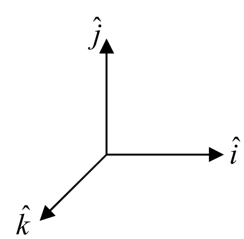
Chapter 1

Tools of Quantum Mechanics

1. Analogy Linear Vector Space

Three-Dimensional Vector Space

A vector in three-dimensional space can be denoted by $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ which have direction and magnitude where the real numbers a_1, a_2, a_3 are responsible of magnitude and unit vector $\hat{i}, \hat{j}, \hat{k}$ are responsible for component along three perpendicular direction.



In another word we can say that vector $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ can be represented in orthonormal basis of unit vector $\hat{i}, \hat{j}, \hat{k}$. The magnitude of vector $\vec{A}$ is denoted by $\left|\vec{A}\right|$, which is equivalent to $\left|\vec{A}\right| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

In this space we define another vector $\vec{B}$ merely changing scalar a_1, a_2, a_3 to b_1, b_2, b_3 and we find another vector $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. So we can construct infinite number of vectors in basis of unit vector $\hat{i},\hat{j},\hat{k}$ which represent three dimensional space. These vectors are example of linear vector spaces.

The vectors in three-dimensional space have given property.

A. Vector Addition

(a) ***Closer Relation***: If vector $\vec{A}$ defined by $\vec{A}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ represented in basis $\hat{i},\hat{j},\hat{k}$ and another vector $\vec{B}$ defined in same basis $\hat{i},\hat{j},\hat{k}$ then addition of $\vec{A}$ and $\vec{B}$ will produce vector $\vec{C}$. Where vector $\vec{C}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ can be represent in same basis vector $\hat{i},\hat{j},\hat{k}$. So $\vec{A}+\vec{B}=\vec{C}$ where $c_1=a_1+b_1, c_2=a_2+b_2$ and $c_3=a_3+b_3$

(b) Commutative Law: $\vec{A}+\vec{B}=\vec{B}+\vec{A}=\vec{C}$

(c) Associative law: $\left(\vec{A}+\vec{B}\right)+\vec{C}=\vec{A}+\left(\vec{B}+\vec{C}\right)$

(d) Existence of Null Vector: A null vector $\vec{O}$ must be part of vector spaces which after vector addition to vector $\vec{A}$ give same vector $\vec{A}$ .In very general view Null vector is definition of origin .

$\vec{A}+\vec{O}=\vec{A}$

B. Scalar multiplication to a vector

Linear Vector spaces also constitute a scalar number $a,b,c...$ which is real number which follow following property with vectors. $a\vec{A}$ means the vector is scaled up or down in magnitude by value a.

$$a\left(\vec{A}+\vec{B}\right)=a\vec{A}+a\vec{B}$$

$$\left(a+b\right)\vec{A}=a\vec{A}=a\vec{A}+b\vec{A}$$

$$ab\vec{A}=ba\vec{A}$$

$$0\vec{A}=\vec{O}$$

C. Scalar Product

The scalar product between two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}.\vec{B}$ where $\vec{A}.\vec{B}=|A|.|B|\cos\theta$ which is equivalent to projection of $\vec{A}$ on vector $\vec{B}$ multiply by magnitude of vector $\vec{B}$. Hence vector $\hat{i},\hat{j}$ and $\hat{k}$ are orthonormal basis vector where $\vec{A}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{B}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{A}.\vec{B}=a_1b_1+a_2b_2+a_3b_3$.

The property of scalar product

(a) Scalar Products Are Commutative: $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

(b) Homogeneous Under Scaling Variable: $\left(a\vec{A}\right).\vec{B} = a\left(\vec{A}.\vec{B}\right) = \vec{A}.\left(a\vec{B}\right)$

(c) Distributive Property: $\vec{A}.\left(\vec{B}+\vec{C}\right) = \vec{A}.\vec{B}+\vec{A}.\vec{C}$

(d) Normal Vector: The magnitude of vector $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is also known as norm of vector can be defined by $|A| = \sqrt{\left(\vec{A}.\vec{A}\right)} = \sqrt{a_1^2 + a_2^2 + a_3^2}$. If we find norm or magnitude of any vector then we can normalize vector and treat them as unit vector $\hat{n}$, where $\hat{n} = \frac{\vec{A}}{|A|} = \frac{\vec{A}}{\sqrt{\vec{A}\cdot\vec{A}}}$

(e) Orthogonal Vector: Two vectors $\vec{A}$ and $\vec{B}$ is said to be orthogonal vector if $\vec{A}.\vec{B} = 0$

D. Triangle Inequality

For given two vectors $\vec{A}$ and $\vec{B}$ $\left|\vec{A}+\vec{B}\right| \le \left|\vec{A}\right| + \left|\vec{B}\right|$

E. Cauchy-Schwarz Inequality

For given two vectors $\vec{A}$ and $\vec{B}$ $\left|\vec{A}.\vec{B}\right| \le \left|\vec{A}\right|.\left|\vec{B}\right|$

If $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then according to Cauchy-schwarz inequality $\sqrt{a_1b_1 + a_2b_2 + a_3b_3} \le \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)\left(\sqrt{b_1^2 + b_2^2 + b_3^2}\right)$

Use of orthonormal basis

If any vector that vector $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ can be represented in orthonormal basis of unit vector $\hat{i}, \hat{j}, \hat{k}$.

Which means the basis vector are treated as unit vectors or normal vectors i.e. $\hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1$ and $\hat{k}.\hat{k} = 1$ as well as they are treated as orthogonal vectors i.e. $\hat{i}.\hat{j} = 0, \hat{j}.\hat{k} = 0$ and $\hat{k}.\hat{i} = 0$.

Due to orthonormal property of basis vectors we can easily defined scalar quantity a_1, a_2 and a_3 by property $a_1 = \hat{i}\cdot\vec{A}$, $a_2 = \hat{j}\cdot\vec{A}$, $a_3 = \hat{k}\cdot\vec{A}$ so vector $\vec{A}$ can be redefined as

$$\vec{A} = \left(\hat{i}\cdot\vec{A}\right)\hat{i} + \left(\hat{j}\cdot\vec{A}\right)\hat{j} + \left(\hat{k}\cdot\vec{A}\right)\hat{k}.$$

In the quantum mechanics these properties of orthonormal basis have huge role to define quantum mechanical state of system.

All these above mention analogies can be beautifully explained by linear vector algebra.