Chapter 1

Tools of Quantum Mechanics

2. Hilbert Space and Dirac Notation

The mathematical concept of a **Hilbert space**, named after David Hilbert, generalizes the notion of Euclidian space. It extends the methods of linear vector algebra and calculus from the two-dimensional Euclidean plane and three-dimensional Euclidean space to spaces with any finite or infinite number of dimensions. A Hilbert space is an abstract vector space possessing the structure of an inner product that allows length and angle to be measured.

Hilbert space H have all property consists of linear vector space. It consists of collection of vector or function $\phi,\xi,\psi...$ represented in Greek letters and scalar numbers $a,b,c...$. The functions $\phi,\xi,\psi...$ are complex function so their complex conjugate can be represented as $\phi^*,\xi^*,\psi^*...$ belong to dual space of Hilbert space H. In quantum mechanics $\phi,\xi,\psi...$ also said to be state of system. $\phi,\xi,\psi...$ and $\phi^*,\xi^*,\psi^*...$ are function of dynamical variable or generalized coordinate of classical system. The scalar numbers $a,b,c...$ are also complex numbers and also belong to duel space.

The function belongs to Hilbert space have followed property.

A. Addition Rule

(a) Closer Relation: If $\psi(x)$ and $\phi(x)$ are belongs to Hilbert Space H then their addition $\phi(x)+\psi(x)=\xi(x)$. $\xi(x)$ will also belong to same Hilbert space H.

(b) Commutation $\phi(x)+\psi(x)=\psi(x)+\phi(x)$

(c) Associative $(\phi(x)+\psi(x))+\xi(x)=\phi(x)+(\psi(x)+\xi(x))$

(d) Existence of Null vector there exists a vector or function represented as O which addition to any function $\psi(x)$ will produce same vector i.e. $\psi(x)+O=\psi(x)$

(e) Existence of inverse: There must be existence of inverse vector of $\psi(x)$ which addition on $\psi(x)$ will produce null vector, i.e. $\psi(x)+(-\psi(x))=O$. Which means $-\psi(x)$ is inverse of $\psi(x)$.

B. Scalar Multiplication

(a) The product of a scalar with a vector gives another vector. In general, if ϕ and ψ are two vectors of the space, any linear combination $a\phi+b\psi$ is also a vector of the space, a and b being scalars.

(b) Distributive with Respect to Addition

$(a+b)\psi(x)=a\psi(x)+b\psi(x)$ and $a(\psi(x)+\phi(x))=a\psi(x)+a\phi(x)$

(c) Associativity with Respect to Multiplication of Scalars

$a(b\psi(x))=ab\psi(x)$

(d) For each element ψ there must exist a unitary scalar I and a zero scalar 0 such that

$I\psi=\psi$ and $0\psi=O$

Scalar Product: If two function $\phi(x)$ and $\psi(x)$ belongs to same Hilbert space H.where x is dynamical variable of system then scalar product can be denoted by (ϕ,ψ). The value of scalar product is

$(\phi,\psi)=\int_{-\infty}^{\infty}\phi^*(x)\psi(x)dx<\infty$. Famously it is also named as inner product. The inner product (ϕ,ψ) must be finite number.

Suppose If two function $\phi(x,y)$ and $\psi(x,y)$ belongs to Hilbert space where dynamical variables are x, y then scalar product is defined as $(\phi,\psi)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi^*(x,y)\psi(x,y)dxdy<\infty$.

Similarly, if dynamical variables are spherical polar coordinate r,θ,ϕ then

$$(\phi,\psi)=\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\phi^*(r,\theta,\phi)\psi(r,\theta\phi)r^2\sin\theta drd\theta d\phi<\infty$$

Properties of Scalar Product

(a) $(\phi,\psi)^*=(\psi,\phi)$

(b) $(\psi,a_1\phi_1+a_2\phi_2)=a_1(\psi,\phi_1)+a_2(\psi,\phi_2)$

(c) $(a_1\phi_1+a_2\phi_2,\psi)=a_1^*(\phi_1,\psi)+a_2^*(\phi_2,\psi)$

(d) $(a_1\phi_1+a_2\phi_2,b_1\psi_1+b_2\psi_2)=a_1^*b_1(\phi_1,\psi_1)+a_1^*b_2(\phi_1,\psi_2)+a_2^*b_1(\phi_2,\psi_1)+a_2^*b_2(\phi_2,\psi_2)$

Square Integrable Function: The function ψ belongs to Hilbert space H with dynamical variable x. Then ψ must be square integrable. The condition for square integrability is

$$(\psi,\psi)=\int_{-\infty}^{+\infty}\psi^*\psi dx=\alpha=\int_{-\infty}^{+\infty}|\psi|^2dx=\alpha,$$

where α is positive constant. So, the norms of function ψ is given by $N=\sqrt{(\psi,\psi)}=\sqrt{\alpha}$

Normalized Function: If norm of any function ψ belong to Hilbert space H is one then function is said to be normalized. Any square integrable function can be normalized when it is divided by its norm which is also known as normalization constant $A=\frac{1}{N}=\frac{1}{\sqrt{(\psi,\psi)}}$. In general normalization condition is $(\psi,\psi)=\int_{-\infty}^{+\infty}\psi^*\psi dx=1\Rightarrow\int_{-\infty}^{+\infty}|\psi|^2dx=1$.

The normalized function can be treated as unit vector.

Orthogonal Function: If two function ϕ and ψ belong to same Hilbert space H. They are said to be orthogonal if scalar product between that function will vanish ie $(\psi,\phi)=0\Rightarrow\int_{-\infty}^{+\infty}\psi^*(x)\phi(x)dx=0$

Orthonormal Function: If two function ϕ_i and ϕ_j belong to same Hilbert space H. They are said to be orthonormal if $(\phi_i,\phi_j)=\int_{-\infty}^{+\infty}\phi_i(x)\phi_j(x)dx=\delta_{ij}=\begin{cases}1,i=j\\0,i\neq j\end{cases}$