Chapter 12

Random Walk Problem

2. One Dimensional Random Walk Problem

Let $x(t)$ denote the position of Brownian Particle at time t given that its position is coincide with the point $x=0, t=0$. To simplify matters, we assume that each molecular impact (which on an average, takes place after time τ^*) causes the particle to jump a small distance l-of constant magnitude –in either positive or negative direction along x axis.

It seems natural to regard the possibilities $\Delta x=+l$ and $\Delta x=-l$ to be equally likely. The probability that particle is found at the point x and t is equal to the probability that ,in series of $n\left(=\frac{t}{\tau^*}\right)$ successive jumps, the particle makes $m\left(=\frac{x}{l}\right)$ more jumps in the positive direction of $x-$axis than in the negative , if n_1 is jumps in the positive direction and if n_2 is jumps in the negative direction .

$$n=n_1+n_2 \text{ and } m=n_1-n_2$$

since the quantities x and t are macroscopic in nature while l and τ^* are microscopic , the numbers n and m are much larger than unity then it can be assumed that they are integral as well.

$n_1 = \frac{1}{2}\left(\frac{n+m}{2}\right)$ $n_2 = \frac{1}{2}\left(\frac{n-m}{2}\right)$ since n_1 and n_2 are integer and if n is even then m is also even and if n is odd then m is also odd. The value of $-n \le m \le n$.

Using Binomial distribution, the probability to getting m step more right than left in n step is given by

$$P_n(m) = {}^nC_{n_1} p^{n_1} q^{n_2} \Rightarrow \frac{\lfloor n}{\lfloor n_1 \lfloor n-n_1}.p^{n_1}.q^{n_2} = \frac{\lfloor n}{\lfloor n_1 \lfloor n_2}.p^{n_1}.q^{n_2}$$

where $n_1 = \frac{1}{2}\left(\frac{n+m}{2}\right)$, $n_2 = \frac{1}{2}\left(\frac{n-m}{2}\right)$ and $p = q = \frac{1}{2}$

$$P_n(m) = \frac{\lfloor n}{\lfloor \frac{n+m}{2} . \lfloor \frac{n-m}{2}}.\left(\frac{1}{2}\right)^n$$

To understand how the distribution forms up, we start by writing out all possible walks for increasing n, for all the possible resulting m by using formula of $P_n(m) = \frac{\lfloor n}{\lfloor \frac{n+m}{2} . \lfloor \frac{n-m}{2}}.\left(\frac{1}{2}\right)^n$

The average value of m is given by $\langle m \rangle = \sum_{m=-n}^{n} mP_n(m) = 0$

The average value of m^2 is given by $\langle m^2 \rangle = \sum_{m=-n}^{n} m^2 P_n(m) = n$

n	m	$\left(\frac{1}{2}\right)(n+m)$	Possible walks	Probability $p_n(m)$
1	-1	0	$\leftarrow$	$\left(\frac{1}{2}\right)$
1	1	1	$\rightarrow$	$\left(\frac{1}{2}\right)$
2	-2	0	$\leftarrow\leftarrow$	$\left(\frac{1}{2}\right)^2$
2	0	1	$\leftarrow\rightarrow; \rightarrow\leftarrow$	$\left(\frac{1}{2}\right)^2 \times 2\times 1$

2	2	2	$\rightarrow\rightarrow$	$\left(\frac{1}{2}\right)^2$
3	-3	0	$\leftarrow\leftarrow\leftarrow$	$\left(\frac{1}{2}\right)^3$
3	-1	1	$\leftarrow\ \leftarrow\rightarrow;\leftarrow\rightarrow\leftarrow;\rightarrow\leftarrow\leftarrow$	$\left(\frac{1}{2}\right)^3\times\frac{3}{1}$
3	1	2	$\leftarrow\rightarrow\rightarrow;\rightarrow\leftarrow\rightarrow;\rightarrow\rightarrow\leftarrow$	$\left(\frac{1}{2}\right)^3\times\frac{3\times 2}{2\times 1}$
3	3	3	$\rightarrow\rightarrow\rightarrow$	$\left(\frac{1}{2}\right)^3 1$
4	-4	0	$\leftarrow\leftarrow\leftarrow\leftarrow$	$\left(\frac{1}{2}\right)^4$
4	-2	1	$\leftarrow\leftarrow\leftarrow\rightarrow;\leftarrow\leftarrow\rightarrow\leftarrow;\leftarrow\rightarrow\leftarrow\leftarrow;\rightarrow\leftarrow\leftarrow\leftarrow;$	$\left(\frac{1}{2}\right)^4\times\frac{4}{1}$
4	0	2	$\leftarrow\leftarrow\rightarrow\rightarrow;\leftarrow\rightarrow\leftarrow\rightarrow;\rightarrow\leftarrow\leftarrow\rightarrow;$ $\rightarrow\leftarrow\rightarrow\leftarrow;\rightarrow\rightarrow\leftarrow\leftarrow;\leftarrow\rightarrow\rightarrow\leftarrow$	$\left(\frac{1}{2}\right)^3\times\frac{4\times 3}{2\times 1}$
4	2	3	$\leftarrow\rightarrow\rightarrow\rightarrow;\rightarrow\leftarrow\rightarrow\rightarrow;\rightarrow\rightarrow\leftarrow\rightarrow;\rightarrow\rightarrow\rightarrow\leftarrow;$	$\left(\frac{1}{2}\right)^4\times\frac{4\times 3\times 2}{3\times 2\times 1}$
4	4	4	$\rightarrow\rightarrow\rightarrow\rightarrow$	$\left(\frac{1}{2}\right)^4$

Hence $t >> \tau^*$ we have net displacement of the particle

$$\langle x(t)\rangle = 0 \text{ and } \langle x^2(t)\rangle = l^2\frac{t}{\tau^*}$$

According, the root-mean-square displacement of the particle is proportional to the square root of the time elapsed :

$$x_{rms} = \sqrt{x^2(t)} = l\sqrt{\left(t/\tau^*\right)} \propto t^{1/2}$$

It should be noted that proportionality of the net overall displacement of the Brownian particle to the square root of the total number of elementary steps.

If successive steps were fully coherent (or else if the motion, were completely predictable and reversible over the time interval t), then net displacement of the Brownian particle would have been proportional to time t .

To obtain an asymptotic form of the function $p_n(m)$, we apply Stirling' s formula, $n! \approx (2\pi n)^{1/2}\left(n/e\right)^n$, to the factorials appearing in

$$\ln P_n(m) \approx \left(n+\frac{1}{2}\right) ln\, n - \frac{1}{2}(n+m+1)\, ln\left\{\frac{1}{2}(n+m)\right\} - \frac{1}{2}(n-m+1)\, ln\left\{\frac{1}{2}(n-m)\right\} - n ln 2 - \frac{1}{2} ln(2\pi)$$

For $m << n$ (which is generally true because $\langle m \rangle = 0$ and $m_{r.m.s} = n^{1/2}$, while $n >> 1$

we obtain

$$p_n(m) \approx \frac{2}{\sqrt{(2\pi n)}} \exp\left(-m^2/2n\right)$$

putting $n = \frac{t}{\tau}$ and $m = \frac{x}{l}$

Talking x to be a continuous variable (and remembering that $P_n(m) \equiv 0$ either for even values of m or for odd values of m, so that in the distribution we may write this above distribution as a form of Gaussian form $p(x)dx = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) dx$

Where $D = \frac{l^2}{2\tau^*}$ where D is define as diffusion coefficient. The statistical distribution of the successive displacements Δx of Brownian particle immersed in water for parameter $(\Delta x)_{rms} = 1.45\mu$