Chapter 1

# Tools of Quantum Mechanics

## 3. Dirac Notation

The function belongs to Hilbert space $\mathrm{H}$ can be represented in term of Dirac notation. It is abstract notation can be efficiently used in quantum mechanics.

If function $\psi(x)$ belongs to Hilbert space $\mathrm{H}$ with dynamical variable $x$ then it can be represented as ket vector as $|\psi\rangle$ pronounced as $\psi$ (shai) ket. Similarly, function $\phi(x)$ can be represented as $|\phi\rangle$ and pronounced as $\phi$ (phai) ket. The $\psi^*(x)$ belongs to duel space of can be represented as Bra vector $\langle\psi|$ pronounced as $\psi$ (shai) Bra. Similarly, function $\phi^*(x)$ can be represented as $\langle\phi|$ and pronounced as $\phi$ (phai) bra.

Mathematically ket vector is equivalent to column vector as $|\psi\rangle \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Bra vector is complex adjoint of ket vector i.e. Row vector represented as $\langle\psi| \equiv \begin{pmatrix} a^* & b^* & c^* \end{pmatrix}$

For example if ket vector is given by $|\psi\rangle = \begin{pmatrix} 2 \\ -3 \\ 3+4i \end{pmatrix}$ then corresponding bra vector is $\langle\psi| = \begin{pmatrix} 2 & -3 & 3-4i \end{pmatrix}$

**A. Addition Rule**

**(a) Closer Relation** If ket$|\psi\rangle$ and ket$|\phi\rangle$ are belongs to Hilbert Space $\mathrm{H}$ then their addition $|\phi\rangle + |\psi\rangle = |\xi\rangle$. Ket$|\xi\rangle$ will also belongs to same Hilbert space $\mathrm{H}$ .

**(b) Commutation** $|\phi\rangle + |\psi\rangle = |\psi\rangle + |\phi\rangle$

**(c) Associative** $\left(|\phi\rangle + |\psi\rangle\right) + |\xi\rangle = |\phi\rangle + \left(|\psi\rangle + |\xi\rangle\right)$

**(d) Existence of Null vector** there exists a vector or function represented as $|O\rangle$ which addition to any function $|\psi\rangle$ will produce same vector i.e. $|\psi\rangle + |O\rangle = |\psi\rangle$

**(e) Existence of inverse** There must be existence of inverse vector of Ket$|\psi\rangle$ which addition on $|\psi\rangle$ will produce null vector . i.e. $|\psi\rangle + \left(-|\psi\rangle\right) = |O\rangle$. Which means $-|\psi\rangle$ is inverse of $|\psi\rangle$.

**B. Scalar Multiplication**

(a) The product of a scalar with a vector gives another vector. In general, if $|\phi\rangle$ and $|\psi\rangle$ are two ketvectors of the space, any linear combination $a|\phi\rangle + b|\psi\rangle$ is also a vector of the space, $a$ and $b$ being scalars.

(b) Distributive with respect to addition

$$(a+b)|\psi\rangle = a|\psi\rangle + b|\psi\rangle \text{ and } a\left(|\psi\rangle + |\phi\rangle\right) = a|\psi\rangle + a|\phi\rangle$$

(c) Associativity with respect to multiplication of scalars

$$a\left(b|\psi\rangle\right) = ab|\psi\rangle$$

(d) For each element $\psi$ there must exist a unitary scalar $I$ and a zero scalar $0$ such that

$$I|\psi\rangle = |\psi\rangle \text{ and } 0|\psi\rangle = |O\rangle$$

**Scalar Product in Dirac Notation**

**Scalar Product:** If two ket vectors $|\phi\rangle$ and$|\psi\rangle$ belongs to same Hilbert space $\mathrm{H}$.where $x$ is dynamical variable of system then scalar product can be denoted by $\langle\phi|\psi\rangle$. The value of scalar product is

$\langle\phi|\psi\rangle=\int_{-\infty}^{\infty}\phi^{*}(x)\psi(x)dx<\infty$. Famously it is also named as inner product. The inner product $\langle\phi|\psi\rangle$ must be finite number.

**Properties of scalar product**

(a) $\langle\phi|\psi\rangle^{*}=\langle\psi|\phi\rangle$

(b) $(|\psi\rangle,a_1|\phi_1\rangle+a_2|\phi_2\rangle)=a_1\langle\psi|\phi_1\rangle+a_2\langle\psi|\phi_2\rangle$

(c) $(a_1|\phi_1\rangle+a_2|\phi_2\rangle,|\psi\rangle)=a_1^{*}\langle\phi_1|\psi\rangle+a_2^{*}\langle\phi_2|\psi\rangle$

(d) $(a_1|\phi_1\rangle+a_2|\phi_2\rangle,b_1|\psi_1\rangle+b_2|\psi_2\rangle)=a_1^{*}b_1\langle\phi_1|\psi_1\rangle+a_1^{*}b_2\langle\phi_1|\psi_2\rangle+a_2^{*}b_1\langle\phi_2|\psi_1\rangle+a_2^{*}b_2\langle\phi_2|\psi_2\rangle$

**Normalized Function:** If norm of any ket vector $|\psi\rangle$ belong to Hilbert space $\mathrm{H}$is one then function is said to be normalized. Any square integrable function can be normalized when it is divided by its norm which is also known as normalization constant $A=\frac{1}{N}=\frac{1}{\sqrt{\langle\psi|\psi\rangle}}$. In general normalization condition is $\langle\psi|\psi\rangle=\int_{-\infty}^{+\infty}\psi^{*}\psi dx=1\Rightarrow\int_{-\infty}^{+\infty}|\psi|^2dx=1$.

**Orthogonal Function:** If two ket vector $|\phi\rangle$and $|\psi\rangle$belong to same Hilbert space $\mathrm{H}$. They are said to be orthogonal if scalar product between that function will vanish ie $\langle\psi|\phi\rangle=0\Rightarrow\int_{-\infty}^{+\infty}\psi^{*}(x)\phi(x)dx=0$

**Orthonormal Function:** If two vectors $|\phi_i\rangle$and $|\phi_j\rangle$belong to same Hilbert space $H$. They are said to be orthonormal if $\langle\phi_i|\phi_j\rangle=\delta_{i,j}\Rightarrow\int_{-\infty}^{+\infty}\phi_i(x)\phi_j(x)dx=\delta_{ij}=\begin{cases}1, i=j\\0, i\neq j\end{cases}$, where $\delta_{i,j}$ is identified as Kronecker delta

**Forbidden State and Direct Product:** If ket$|\phi\rangle$ and ket$|\psi\rangle$ belongs to same Hilbert space $\mathrm{H}$ then direct product $|\phi\rangle\otimes|\psi\rangle$ and $\langle\phi|\otimes\langle\psi|$ are meaningless , they are void and Nonsensical. If ket$|\phi\rangle$ and ket$|\psi\rangle$ belongs to different Hilbert space $\mathrm{H}$ then direct product $|\phi\rangle\otimes|\psi\rangle$ and $\langle\phi|\otimes\langle\psi|$ are identified as direct product .

**Triangle Inequality:** If ket$|\phi\rangle$ and ket$|\psi\rangle$ belongs to same Hilbert space $\mathrm{H}$

$$\sqrt{\langle\psi+\phi|\psi+\phi\rangle}\leq\sqrt{\langle\psi|\psi\rangle}+\sqrt{\langle\phi|\phi\rangle}$$

**Cauchy Swartz inequality** If ket$|\phi\rangle$ and ket$|\psi\rangle$ belongs to same Hilbert space $\mathrm{H}$ then

$$\left|\langle\psi|\phi\rangle\right|^2 \le \left(\langle\psi|\psi\rangle\right)\left(\langle\phi|\phi\rangle\right)$$

**Proof:** $0 \le \left\||\phi\rangle - \lambda|\psi\rangle\right|^2$ where $\lambda = \dfrac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}$ so $\lambda^* = \dfrac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}$

$$0 \le \left\||\phi\rangle - \lambda|\psi\rangle\right|^2 \Rightarrow 0 \le \left(|\phi\rangle - \lambda|\psi\rangle, |\phi\rangle - \lambda|\psi\rangle\right)$$

$$\Rightarrow 0 \le \langle\phi|\phi\rangle - \lambda\langle\phi|\psi\rangle - \lambda^*\langle\psi|\phi\rangle + \lambda^*\lambda\langle\psi|\psi\rangle$$

$$\Rightarrow 0 \le \langle\phi|\phi\rangle - \frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}.\langle\phi|\psi\rangle - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\langle\psi|\phi\rangle + \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}.\frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}\langle\psi|\psi\rangle$$

$$\Rightarrow \langle\phi|\phi\rangle \le -\frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}.\langle\phi|\psi\rangle - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\langle\psi|\phi\rangle + \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}.\frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}\langle\psi|\psi\rangle$$

$$\Rightarrow \langle\phi|\phi\rangle \le -\frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}.\langle\phi|\psi\rangle - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\langle\psi|\phi\rangle + \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}.\langle\phi|\psi\rangle$$

$$\Rightarrow \langle\phi|\phi\rangle \le -\frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}.\langle\phi|\psi\rangle \Rightarrow \langle\phi|\phi\rangle\langle\psi|\psi\rangle \le -\langle\psi|\phi\rangle.\langle\phi|\psi\rangle$$

$$\Rightarrow \langle\phi|\phi\rangle\langle\psi|\psi\rangle \ge \langle\psi|\phi\rangle.\langle\phi|\psi\rangle \Rightarrow \left|\langle\psi|\phi\rangle\right|^2 \le \left(\langle\psi|\psi\rangle\right)\left(\langle\phi|\phi\rangle\right)$$

**Linear Independency**

The vector $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, ...., |\phi_N\rangle$ are said to be linear independent if we find the coefficient $c_1, c_2, c_3, ...., c_N$ which satisfy the equation. $c_1|\phi_1\rangle + c_2|\phi_2\rangle + c_3|\phi_3\rangle + .... + c_N|\phi_N\rangle = 0$ and get the unique solution as

$c_1 = c_2 = c_3 = ..... = c_N = 0$ then $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, ...., |\phi_N\rangle$ is said to be linearly independent.

The *dimension* of a vector space is given by the *maximum number* of linearly independent vectors the space can have. For instance, if the maximum number of linearly independent vectors a space has is $N$ $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, ...., |\phi_N\rangle$ this space is said to be $N$ - dimensional. In this $N$ -dimensional vector space, any vector $|\psi\rangle$ can be expanded as a linear combination i.e. $|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + c_3|\phi_3\rangle + .... + c_N|\phi_N\rangle$

**Relation between orthogonal and linearly independent vectors**

Every orthogonal vector is linearly independent but it is not necessary that every linear independent vector is orthogonal.

In quantum mechanics any vector $|\psi\rangle$ can be represented in orthonormal basis of vector $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, ...|\phi_i\rangle....|\phi_j\rangle....|\phi_N\rangle$ as

$$|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle + c_3|\phi_3\rangle + ..c_i|\phi_i\rangle...c_j|\phi_j\rangle..... + c_N|\phi_N\rangle \Rightarrow |\psi\rangle = \sum_{i=1}^{N} c_i|\phi_i\rangle$$

Hence it is given $\langle\phi_i|\phi_j\rangle = \delta_{i,j}$ so the term $c_i = \langle\phi_i|\psi\rangle$ and $c_j = \langle\phi_j|\psi\rangle$

So $|\psi\rangle = \sum_{i=1}^{N} c_i|\phi_i\rangle = \sum_{i=1}^{N}(\langle\phi_i|\psi\rangle)|\phi_i\rangle$ which means the coefficient $c_i$ is scalar product between component basis vector $|\phi_i\rangle$ and vector $|\psi\rangle$. This trick can be only applied if vectors can be represented in orthonormal basis vector. Here basis vectors $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, \ldots |\phi_i\rangle \ldots |\phi_j\rangle \ldots |\phi_N\rangle$ are analogically orthonormal unit vectors in Euclidian space.

**Example:** If $\psi(x) = \begin{cases} 0, & x<0 \\ A\exp\left(-\frac{x}{a}+ikx\right), & x>0 \end{cases}$ where $a, k$ are positive constant, then find the value of $A$ such that $\psi(x)$ is normalized.

**Solution:** $\psi(x) = \begin{cases} 0, & x<0 \\ A\exp\left(-\frac{x}{a}+ikx\right), & x>0 \end{cases}$ and $\psi^*(x) = \begin{cases} 0, & x<0 \\ A\exp\left(-\frac{x}{a}-ikx\right), & x>0 \end{cases}$

For normalization $(\psi,\psi)=1 \Rightarrow \int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = 1$

$$\int_{-\infty}^{0} 0dx + \int_0^{\infty} A^* \exp\left(-\frac{x}{a}-ikx\right)A\exp\left(-\frac{x}{a}+ikx\right)dx = 1 \Rightarrow |A|^2\int_0^{\infty}\exp\left(-\frac{2x}{a}\right)dx = 1$$

$\Rightarrow A = \sqrt{\frac{2}{a}}$ so $\psi(x) = \begin{cases} 0, & x<0 \\ \sqrt{\frac{2}{a}}\exp\left(-\frac{x}{a}+ikx\right), & x>0 \end{cases}$ is normalized function .

**Example:** Assume two function $\phi(x)$ and $\psi(x)$, which is defined as $\phi(x) = \begin{cases} a, & -1\le x\le 1 \\ 0, & otherwise \end{cases}$ and $\psi(x) = \begin{cases} bx+c, & -1\le x\le 1 \\ 0, & otherwise \end{cases}$. Find the condition on $a, b, c$ (real numbers) such that $\phi(x)$ and $\psi(x)$ are orthonormal vector.

**Solution:** Condition such that $\phi(x)$ is normalized $(\phi,\phi) = \int_{-\infty}^{\infty}\phi^*\phi dx = 1 \Rightarrow \int_{-1}^{1} a^2 dx = 1 \Rightarrow a = \frac{1}{\sqrt{2}}$

Condition such that $\phi(x)$ and $\psi(x)$ are orthogonal

$$(\phi,\psi) = 0 \Rightarrow \int_{-\infty}^{\infty}\phi^*\psi dx = 0 \Rightarrow \int_{-1}^{1}\frac{1}{\sqrt{2}}(bx+c)dx = 0 \Rightarrow c = 0$$

Condition for $\psi(x)$ is normalized $(\psi,\psi) = 1 \Rightarrow \int_{-\infty}^{\infty}\psi^*\psi dx = 0 \Rightarrow \int_{-1}^{1}(bx)^2 dx = 1 \Rightarrow b = \sqrt{\frac{3}{2}}$

So, $\phi(x) = \begin{cases} \frac{1}{\sqrt{2}}, & -1\le x\le 1 \\ 0, & otherwise \end{cases}$ and $\psi(x) = \begin{cases} \sqrt{\frac{3}{2}}x, & -1\le x\le 1 \\ 0, & otherwise \end{cases}$ are orthonormal functions.

**Example:** function $\phi(x)=\exp(-\alpha x^2)$, where $-\infty<x<+\infty$, and $\psi(x)=x\exp(-\alpha x^2)$ where $-\infty<x<\infty$ belong to Hilbert space with variable $x$

(a) Find norms of $\phi(x)$ and $\psi(x)$ write expression of normalized $\phi(x)$ and $\psi(x)$.

(b) Prove that $\phi(x)$ and $\psi(x)$ are orthogonal.

Use the integration $\int_0^\infty x^n\exp(-\beta x^2)\,dx=\frac{1}{2}\cdot\frac{1}{\beta^{\frac{n+1}{2}}}\left|\overline{\frac{n+1}{2}}\right.$

**Solution:** (a) Norms of $\phi(x)$ is

$$N_1=\sqrt{(\phi,\phi)},\ (\phi,\phi)=\int_{-\infty}^{\infty}\phi^*(x)\phi(x)\,dx\Rightarrow\int_{-\infty}^{\infty}\exp(-2\alpha x^2)dx\Rightarrow 2\times\frac{1}{2}\times\left(\frac{1}{2\alpha}\right)^{1/2}\left|\overline{\frac{1}{2}}\right.=\left(\frac{1}{2\alpha}\right)^{1/2}\sqrt{\pi}$$

$N_1=\left(\frac{\pi}{2\alpha}\right)^{1/4}$, so normalized constant $A_1=\frac{1}{N_1}\Rightarrow A_1=\left(\frac{2\alpha}{\pi}\right)^{1/4}$

Normalized function $\phi(x)=\left(\frac{2\alpha}{\pi}\right)^{1/4}\exp(-\alpha x^2)$

Norms of $\psi(x)$ is $N_2=\sqrt{(\psi,\psi)}$ ,

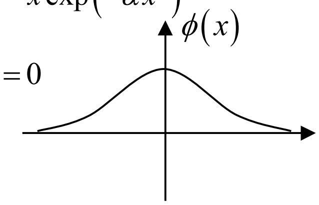
$$(\psi,\psi)=\int_{-\infty}^{\infty}\psi^*(x)\psi(x)\,dx\Rightarrow\int_{-\infty}^{\infty}x^2\exp(-2\alpha x^2)dx\Rightarrow 2\times\frac{1}{2}\times\left(\frac{1}{2\alpha}\right)^{3/2}\left|\overline{\frac{3}{2}}\right.=\left(\frac{1}{2\alpha}\right)^{3/2}.\frac{1}{2}.\sqrt{\pi}$$

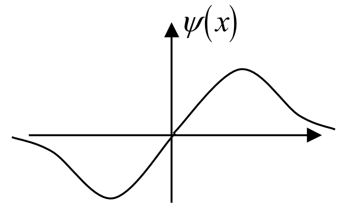
$N_2=\left(\frac{1}{2}\right)^{1/2}\left(\frac{\pi^{1/4}}{(2\alpha)^{3/4}}\right)$ so normalized constant $A_2=\frac{1}{N_2}\Rightarrow A_1=2^{1/2}\left(\frac{(2\alpha)^3}{\pi}\right)^{1/4}$

Normalized function $\psi(x)=2^{1/2}\left(\frac{(2\alpha)^3}{\pi}\right)^{1/4}x\exp(-\alpha x^2)$

(b) If $\phi$ and $\psi$ are orthogonal, then $(\phi,\psi)=0$

$$\Rightarrow A_1.A_2\int_{-\infty}^{+\infty}e^{-\alpha x^2}xe^{-\alpha x^2}dx=0$$




**Example:** (a) If $|\psi\rangle=A\begin{pmatrix}1\\ i\\ 0\end{pmatrix}$, Find the value of $A$ such that $|\psi|$ is normalized.

**Solution:** (a) If ket vector is $|\psi\rangle=A\begin{pmatrix}1\\ i\\ 0\end{pmatrix}$ then bra vector $\langle\psi|=A^*(1\ \ -i\ \ 0)$

For normalization condition $\langle\psi|\psi\rangle=1$

$$\Rightarrow A^*(1\ \ -i\ \ 0).A\begin{pmatrix}1\\ i\\ 0\end{pmatrix}=1\Rightarrow|A|^2(1+1+0)=1\Rightarrow|A|^2=\frac{1}{2}\Rightarrow A=\frac{1}{\sqrt{2}}$$

**Example:** If $|\psi\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$ it is also given $\langle\phi_i|\phi_j\rangle = \delta_{i,j}$ find $\langle\psi|\psi\rangle$ and $|\psi|^2$.

**Solution:** If ket$|\psi\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$

$\langle\psi| = a_1^*\langle\phi_1| + a_2^*\langle\phi_2|$

$\langle\psi|\psi\rangle = a_1^*a_1\langle\phi_1|\phi_1\rangle + a_1^*a_2\langle\phi_1|\phi_2\rangle + a_2^*a_1\langle\phi_2|\phi_1\rangle + a_2^*a_2\langle\phi_2|\phi_2\rangle$

Hence $\langle\phi_i|\phi_j\rangle = \delta_{i,j}$ so $\langle\phi_1|\phi_1\rangle = 1, \langle\phi_2|\phi_2\rangle = 1, \langle\phi_1|\phi_2\rangle = 0, \langle\phi_2|\phi_1\rangle = 0$

$\langle\psi|\psi\rangle = a_1^*a_1 + a_2^*a_2 = |a_1|^2 + |a_2|^2$ which is scalar number

$|\psi|^2 = \psi^*\psi$, $\psi = a_1\phi_1 + a_2\phi_2, \psi^* = a_1^*\phi_1^* + a_2^*\phi_2^*$

$\psi^*\psi = \left(a_1^*\phi_1^* + a_2^*\phi_2^*\right)\left(a_1\phi_1 + a_2\phi_2\right)$

$|\psi|^2 = a_1^*a_1\phi_1^*\phi_1 + a_1^*a_2\phi_1^*\phi_2 + a_2^*a_1\phi_2^*\phi_1 + a_2^*a_2\phi_2^*\phi_2 = |a_1|^2|\phi_1|^2 + a_1^*a_2\phi_1^*\phi_2 + a_2^*a_1\phi_2^*\phi_1 + |a_2|^2|\phi_2|$

$|\psi|^2 = |a_1|^2|\phi_1|^2 + |a_2|^2|\phi_1|^2 + 2\,\mathrm{Re}\left(a_1^*a_2\phi_1^*\phi_2\right)$ which is a function

**Example:** If $|\psi_1\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$ and $|\psi_2\rangle = b_1|\phi_1\rangle + b_2|\phi_2\rangle$ it is given $\langle\phi_i|\phi_j\rangle = \delta_{i,j}$ find the condition such that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthonormal.

**Solution:** Condition such that $|\psi_1\rangle$ is normalized $\langle\psi_1|\psi_1\rangle = 1$

$\langle\psi_1|\psi_1\rangle = 1 \Rightarrow a_1^*a_1\langle\phi_1|\phi_1\rangle + a_2^*a_2\langle\phi_2|\phi_2\rangle + a_1^*a_2\langle\phi_1|\phi_2\rangle + a_2^*a_1\langle\phi_2|\phi_1\rangle = 1$ if $\langle\phi_i|\phi_j\rangle = \delta_{i,j}$

$\langle\phi_1|\phi_1\rangle = 1, \langle\phi_2|\phi_2\rangle = 1, \langle\phi_1|\phi_2\rangle = 0, \langle\phi_2|\phi_1\rangle = 0$

$\langle\psi_1|\psi_1\rangle = 1 \Rightarrow |a_1|^2 + |a_2|^2 = 1$

$\langle\psi_2|\psi_2\rangle = 1 \Rightarrow b_1^*b_1\langle\phi_1|\phi_1\rangle + b_2^*b_2\langle\phi_2|\phi_2\rangle + b_1^*b_2\langle\phi_1|\phi_2\rangle + b_2^*b_1\langle\phi_2|\phi_1\rangle = 1$

if $\langle\phi_i|\phi_j\rangle = \delta_{i,j}, \langle\phi_1|\phi_1\rangle = 1, \langle\phi_2|\phi_2\rangle = 1, \langle\phi_1|\phi_2\rangle = 0, \langle\phi_2|\phi_1\rangle = 0$

$$b_1^*b_1 + b_2^*b_2 = 1 \Rightarrow |b_1|^2 + |b_2|^2 = 1$$

$\langle\psi_1|\psi_2\rangle = 0 \Rightarrow a_1^*b_1\langle\phi_1|\phi_1\rangle + a_2^*b_2\langle\phi_2|\phi_2\rangle + a_1^*b_2\langle\phi_1|\phi_2\rangle + a_2^*b_1\langle\phi_2|\phi_1\rangle = 0$

if $\langle\phi_i|\phi_j\rangle = \delta_{i,j}, \langle\phi_1|\phi_1\rangle = 1, \langle\phi_2|\phi_2\rangle = 1, \langle\phi_1|\phi_2\rangle = 0, \langle\phi_2|\phi_1\rangle = 0$

$\langle\psi_1|\psi_2\rangle = 0 \Rightarrow a_1^*b_1 + a_2^*b_2 = 0$

Similarly, $\langle\psi_2|\psi_1\rangle = 0 \Rightarrow b_1^*a_1 + b_2^*a_2 = 0$

**Example:** The vector in three-dimensional space is defined as

$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $|\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $|\phi_3\rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Prove that they are linearly independent

For $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ linearly independent $c_1|\phi_1\rangle + c_2|\phi_2\rangle + c_3|\phi_3\rangle = 0$

$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$ solving the equation we get $c_1 = 0$, $c_2 + c_3 = 0$, $c_2 - c_3 = 0$

Which has unique solution $c_1 = c_2 = c_3 = 0$ so they are linearly independent

**Example:** Prove that $\phi_1 = x, \phi_2 = x^2, \phi_3 = x^3$ are linearly independent

$$c_1\phi_1 + c_2\phi_2 + c_3\phi_3 = 0 \Rightarrow c_1 x + c_2 x^2 + c_3 x^3 = 0$$

equating coefficient, we get $c_1 = c_2 = c_3 = 0$ which ensure that $x, x^2, x^3$ are linearly independent

**Example:** Prove that $\phi_1 = \exp - x^2, \phi_2 = x\exp(-x^2)$

$$c_1\phi_1 + c_2\phi_2 = 0 \Rightarrow \exp(-\alpha x^2)[c_1 x + c_2] = 0 \text{ hence } \exp(-\alpha x^2) \neq 0$$

$[c_1 x + c_2] = 0 \Rightarrow c_1 = c_2 = 0$ which ensure $\phi_1 = \exp - x^2, \phi_2 = x\exp(-x^2)$ are linearly independent.

**Example:** (a) Prove that vector $|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent but not orthogonal

(b) Prove that vector $|\phi_1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are orthogonal. Also check their linearly independency.

**Solution:** (a) $c_1|\phi_1\rangle + c_2|\phi_2\rangle = 0 \Rightarrow c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

$\Rightarrow c_1 + c_2 = 0$ and $c_2 = 0$ which ensure $c_1 = 0, c_2 = 0$ that means $|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are linearly independent.

Now we check orthogonal relation $\langle \phi_1 | \phi_2 \rangle = 0$

$\Rightarrow \langle \phi_1 | \phi_2 \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \neq 0$ which means $|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are not orthogonal

So it is not necessary that linear independent vectors are orthogonal.

(b) If $|\phi_1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we check orthogonal relation $\langle \phi_1 | \phi_2 \rangle = 0$

$\Rightarrow \langle \phi_1 | \phi_2 \rangle = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \times 1 + 1 \times (-1) = 0$, which means $|\phi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are orthogonal.

So, if vectors are orthogonal then they must be linearly independent.

Let's check there independency

$$c_1 |\phi_1\rangle + c_2 |\phi_2\rangle = 0 \Rightarrow c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$\Rightarrow c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ which ensure $c_1 = 0, c_2 = 0$ that means $|\phi_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent.