Chapter 11
Phase Transition and Low Temperature Physics

4. Landau Theory of Phase Transition

Landau has given a systematic formulation of the theory of phase transition. Consider a system at constant volume and temperature with Helmholtz free energy (F = U − TS) minimum at equilibrium. We suppose that the system can be described by a single order parameter ξ (The Greek xi) which might be the magnetization in a ferromagnetic system, the dielectric polarization, the fraction of superconducting electrons in a superconductor. In thermal equilibrium, the order parameter will have a certain value $\xi_0(t)$, where t is temperature. We define the Landau free energy function $F_L(\xi,t) = U\ (\xi,t) - rS(\xi,T)$ where the energy U and entropy S are taken when the order parameter has the value ξ. The equilibrium value $\xi_0(t)$ is the value of ξ that makes F_L minimum at a given t and actual. Helmholtz free energy $F(t)$ of the system at t is equal to that minimum.

$$F(t) = F_L(\xi_0,t) \le F_L(\xi,t) \text{ if } (\xi \ne \xi_0)$$

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Plotted as a function of ξ for constant t, the Landau free energy may have more than one minimum. The lowest of these determines the equilibrium state. In a first order transition another minimum becomes the lowest minimum as t is increased.

We restrict ourselves to systems in which the Landau function is an even function of ξ in the absence of applied fields. We also assume that $F_L(\xi,t)$ is a sufficiently well-behaved function of ξ that can be expanded in a power series in ξ. For an even function of ξ, let

$$F_L(\xi,t) = g_0(t) + \frac{1}{2}g_2(t)\xi^2 + \frac{1}{4}g_4(t)\xi^4 + \frac{1}{6}g_6(t)\xi^6$$

The entire temperature dependence of $F_L(\xi,t)$ is contained in the expansion coefficients g_0, g_2, g_4, g_6. The simplest example of a phase transition occurs when $g_2(t)$ changes sign at a temperature t_0 with positive g_4 and higher terms negligible. For simplicity, we take $g_2(t) = (t - t_0)\alpha$, over the temperature range of interest, and we take g_4 as constant in that range. With these idealizations

Figure: Temperature dependence of the free energy for an idealized second order

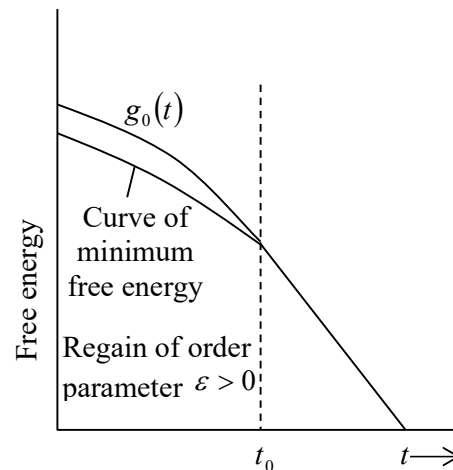$$F_L(\xi,t) = g_0(t) + \frac{1}{2}\alpha(t - t_0)\xi^2 + \frac{1}{4}g_4\xi^4$$

The equilibrium value of ξ is found at the minimum of $F(\xi,t)$ with respect to ξ. That is,

$$\left(\frac{\partial F_L}{\partial \xi}\right) = (t - t_0)\alpha\xi + g_4\xi^3 = 0$$

or $\quad \xi\left[(t - t_0)\alpha + g_4\xi^2\right] = 0$

which has the roots $\xi = 0$ and $\xi^2 = (t_0 - t)\dfrac{\alpha}{g_4}$. With α and g_4 positive, the root $\xi = 0$ corresponds to the minimum of the free energy function F_L at temperature above t_0. Here the free function F_L at the temperature above t_0. Here the free energy is $F(t) - g_0(t)$. The other root $(\alpha/g_4)(t_0 - t)$ corresponds to the minimum of the free energy at temperature below t_0. Here the Helmholtz free energy.

$$F(t) = g_0(t) - \frac{\alpha^2}{4g_4}(t - t_0)^2$$

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(b) The relation between pressure (P) and density (ρ) at critical temperature as

$$(P - P_C)(\rho - \rho_C)^\delta, \ T = T_C$$

It appears that these indices have the same value on either side of T_C. However, one can make a distinction between the indices for values $T > T_C$ and those for $T < T_C$.

Dimensional Analysis:

The scaling hypothesis deals with how various quantities change under a change of length scale. Important scaling laws can be derived by assuming that, near the critical point, the correlation length ξ is the only characteristic length of the system, in terms of which all other lengths must be measured.

Another way is to assume that a quantity of dimension (length) is proportional to ξ^{-d} near the critical point. Suppose a quantity G/kT is dimensionless. Therefore, $g = G/kTV$ is of dimension (length) $-d$ and we indicate this by

$$[g] = L^{-d}$$

Normalizing the correlation function, according to equation

$$\gamma(r) \underset{t \to 0}{\longrightarrow} r^{-P} e^{-r/\xi} \ (\text{Ornsteia} - \text{Zernike form})$$

we have $[\gamma(x)] = L^{2-d-n}$

By definition, this has the same dimension as $\langle m(0) \rangle^2$. Hence

$$[M/V] = L^{(2-d-n)/2}$$

where $m(O) = $ order parameter density and M = order parameter.

According to the Fluctuation-Dissipation theorem, we have

$$[KT_x] = L^{2-\eta}$$

The dimension of the conjugate field can be obtained from the relation

$$M = -\partial G/\partial H [H/kT] = [g]/[M/V] = L^{(2+d-\eta)/2}$$

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These results are summarized in following table:

Function F	Dimension D	Exponent Scaling Hypothesis	Definition
G/kTV	d	vd	$2-\alpha$
M/V	$(d-2-\eta)/2$	$-v(2-d-\eta)/2$	β
KT_x	$\eta-2$	$-v(2-\eta)$	$-\gamma$
H/kT	$(2+d-\eta)/2$	$v(2+d-\eta)/2$	$\beta\delta$
		$^\alpha[F]=(length)^{-d}$	

Table: Dimension and Exponents

The exponent of L is denoted by $-d$ and d is called the "dimension" [The (-)ve sign is introduced such that a quantity changes by a factor b^d when the unit of length is increased by a factor b]. Now replace the length L in the formulas above by ξ. Using $\xi \sim t^{-\upsilon}$ we get all the critical exponents. Comparing them with their definitions leads to the relations

$$2 - \alpha = vd$$
$$\beta = -v(2-d-\eta)/2,$$
$$\gamma = v(2-\eta)$$

and
$$\beta\delta = v(d+d-\eta)/2$$

The first is the Josephson law, and the third is the Fisher law. By subtraction and addition, we derive from the second and the fourth the Rushbrook and the Widom laws.

density difference,

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