Chapter 12

Random Walk Problem

4. Langevin Equation

In statistical physics, a **Langevin equation** given by Paul Langevin is a stochastic differential describing the time evolution of a subset of the degrees of freedom. These degrees of freedom typically are collective (macroscopic) variables changing only slowly in comparison to the other (microscopic) variables of the system. The fast (microscopic) variables are responsible for the stochastic nature of the Langevin equation.

Differential equation envolving Random external force F_e and restraining force similar to frictional force which is proportional to velocity

The equation of motion is given by $M\ddot{\vec{r}} + \mu\dot{\vec{r}} = \vec{F}_e$ where M is mass of center of mass and μ is friction coefficient .

Hence forces are random then $\langle F_e \rangle = 0$ and $\langle \vec{r}.\vec{F}_e \rangle = 0$

Let us suppose that the force F_e in the Langevin equation is the random force due to collisions with the gas molecules only, and attempt to deduce the time variation of the average square displacement of the body, $\langle r^2 \rangle$. Our aim is to show that this system, described by Eq $M\ddot{\vec{r}} + \mu\dot{\vec{r}} = \vec{F}_e$. has a solution that describes Brownian motion. We expect, therefore, $\langle r^2 \rangle$ to be proportional to the time.

Since the solution involves analytic arguments along with statistical arguments, As we shall discuss it in detail. As we are interested in the change of the magnitude Here we will restrict ourselves to a few comments concerning such equation:

(a) To any given sequence, F_e, corresponds a particular solution;

(b) The solution corresponding to any particular sequence of the random force is of little interest;

(c) A quantity can be significant only if it is not strongly dependent on the particular sequence;

(d) Such a quantity can be calculated by averaging over all "acceptable" sequences, just because it is insensitive.

$\langle r^2 \rangle$, we first obtain an equation for r^2: so taking scalar product with $\vec{r}$ on both side and $M\ddot{\vec{r}} + \mu\dot{\vec{r}} = \vec{F}_e$ reduce to $M\vec{r}.\ddot{\vec{r}} + \mu\vec{r}.\dot{\vec{r}} = \vec{r}.\vec{F}_e$ and using the identity $\frac{d}{dt}r^2 = 2r.\frac{dr}{dt}$, and $\frac{d^2}{dt^2}r^2 = 2r \cdot \frac{d^2r}{dt^2} + 2\dot{r}^2$ one will get differential equation

$$\frac{1}{2}M\frac{d^2r^2}{dt^2} + \frac{1}{2}\mu\frac{dr^2}{dt} - M\dot{r}^2 = r \cdot F_e.$$

Note that above Equation. goes beyond the familiar context of differential equations. Beside functions of t and their derivatives, It is a stochastic differential equation.

We obtain a differential equation for $\langle r^2 \rangle$:

$$\frac{1}{2}M\frac{d^2}{dt^2}\langle r^2 \rangle + \frac{1}{2}\mu\frac{d}{dt}\langle r^2 \rangle - 2\left\langle \frac{1}{2}M\upsilon^2 \right\rangle = \langle \vec{r}.\vec{F}_e \rangle,$$

$$\frac{1}{2}M\frac{d^2}{dt^2}\langle r^2 \rangle + \frac{1}{2}\mu\frac{d}{dt}\langle r^2 \rangle - 2\left\langle \frac{1}{2}M\upsilon^2 \right\rangle = 0,$$

Where v was substituted for $\dot{r}$.

The last average on the left-hand side can be evaluated, at equilibrium, by the equipartition principle. Every degree of freedom, for each dimension of space, is assigned an energy of $\frac{1}{2}kT$.

The last term is therefore DkT, where D denotes the number of dimensions of space.

Thus the equation becomes $M\ddot{u}+\mu\dot{u}=2DkT$, where we denoted $\langle r^2\rangle$ by u. This equation can be fully solved. The initial conditions are chosen to be $r(t=0)=0$, namely the origin of the coordinate system of each body in the ensemble is chosen as its position at $t=o$. In this case $u(t=0)=\dot{u}(t=0)=0$

$$u(t)=\frac{2DkT}{\mu}\left[t+\theta\left(e^{-t|\theta}-1\right)\right],$$

where $\theta=\frac{M}{\mu}$. Let us inquire what happens to the body a very short and a very long time after the initiation of its motion, and compare to our physical institution. Short and long times must be measured with respect to a characteristic time appearing in the problem. In our case this characteristic time is θ. That is, at short times $t\langle\langle\theta$, it is possible to expand the exponential in Eq. $u(t)=\langle r^2\rangle\approx\frac{DkT}{M}t^2$..