Chapter 1

Tools of Quantum Mechanics

6. Probability Density

If x is any random variable then any function $f(x)$ can be treated as probability density if $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ means Area under probability density curve $f(x)$ is 1

Probability such that system has value of x between a to b

i.e. $P(a < x < b) = \dfrac{\int_a^b f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}$

average value of x if probability density function is $f(x)$ is given by $\langle x \rangle = \dfrac{\int_{-\infty}^{\infty} x f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}$

average value of x^2 if probability density function is $f(x)$ is given by $\langle x^2\rangle = \dfrac{\int_{-\infty}^{\infty} x^2 f(x)dx}{\int_{-\infty}^{\infty} f(x)dx}$

standard deviation is measurement of dispersion given by Δx i.e. $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ the variance is $(\Delta x)^2 \geq 0$ variance so for any random variable process $\langle x^2\rangle \geq \langle x\rangle^2$

Example: The probability density is defined as $\begin{cases} f(x) = x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$

(a) Find the Probability such that $0 < x < \frac{1}{2}$.

(b) Find $\langle x\rangle$, $\langle x\rangle^2$ and Δx.

Solution: (a) $p\left(0 \leq x \leq \frac{1}{2}\right) = \dfrac{\int_0^{1/2} f(x)dx}{\int_0^1 f(x)dx} = \dfrac{\int_0^{1/2} xdx}{\int_0^1 xdx} = \frac{1}{4}$

$$\langle x\rangle = \frac{\int_0^1 xf(x)dx}{\int_0^1 f(x)dx} = \frac{\int_0^1 xxdx}{\int_0^1 xdx} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \text{ and } \langle x^2\rangle = \frac{\int_0^1 x^2 f(x)dx}{\int_0^1 f(x)dx} = \frac{\int_0^1 x^2 xdx}{\int_0^1 xdx} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{2}{4} = \frac{1}{2}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \Delta x = \sqrt{\frac{1}{2} - \frac{4}{9}} = \sqrt{\frac{1}{18}}$$

Example: $f(x) = \begin{cases} x\exp\left(-\frac{x}{\lambda}\right), & for\ x \geq 0 \\ 0 \quad , & \text{otherwise} \end{cases}$, where $\lambda > 0$

(a) Find then the probability that system has value $x \geq \lambda$ assume $\lambda > 0$

(b) Find $\langle x\rangle, \langle x^2\rangle$ and most probable value of x

(c) Find the most probable value of x.

Solution: (a) $p(x \geq \lambda) = \dfrac{\int_\lambda^\infty \exp(-\frac{x}{\lambda})dx}{\int_0^\infty \exp(-\frac{x}{\lambda})dx} = \frac{1}{e}$

(b) $\langle x\rangle = \dfrac{\int_{-\infty}^{\infty} xf(x)dx}{\int_{-\infty}^{\infty} f(x)dx} = \dfrac{\int_0^\infty x.x\exp\left(-\frac{x}{\lambda}\right)dx}{\int_0^\infty x\exp\left(-\frac{x}{\lambda}\right)dx} = \dfrac{\lambda^3 \lceil 3}{\lambda^2 \lceil 2} = 2\lambda$

$$\langle x^2\rangle=\frac{\int_{-\infty}^{\infty}x^2f(x)dx}{\int_{-\infty}^{\infty}f(x)\,dx}=\frac{\int_0^{\infty}x^2.x\exp\left(-\frac{x}{\lambda}\right)dx}{\int_0^{\infty}x\exp\left(-\frac{x}{\lambda}\right)dx}=\frac{\lambda^4\lceil 4}{\lambda^2\lceil 2}=6\lambda^2$$

(c) For most probable value $\frac{df}{dx}=0\Rightarrow\frac{dx\exp\left(-\frac{x}{\lambda}\right)}{dx}=0$

$$\Rightarrow\exp\left(-\frac{x}{\lambda}\right)+x\left(-\frac{1}{\lambda}\right)\exp\left(-\frac{x}{\lambda}\right)=0\Rightarrow\exp\left(-\frac{x}{\lambda}\right)\left(1-\frac{x}{\lambda}\right)=0$$

Hence $\exp\left(-\frac{x}{\lambda}\right)\neq 0$ so $\left(1-\frac{x}{\lambda}\right)=0\Rightarrow x=\lambda$ we can check $\left.\frac{d^2f}{dx^2}\right|_{x=\lambda}<0$ so $x=\lambda$ is $f(x)$ will be maximum $f(\lambda)=\frac{\lambda}{e}$