TIFR-2023 [Solution]

Section A

Q1. A surface is given by $4x^2y - 2xy^2 + 3z^3 = 0$

Which of the following is a vector normal to it at the point (2,3,1)?

(a) $30\hat{i} - 8\hat{j} + 9\hat{k}$ (b) $15\hat{i} - 4\hat{j} + 9\hat{k}$ (c) $30\hat{i} + 8\hat{j} - 9\hat{k}$ (d) $30\hat{i} - 8\hat{j} - 9\hat{k}$

Topic-Mathematical Physics

Sub topic: Vector Analysis

Ans.: (a)

Solution: $\nabla f(x,y,z)_{(2,3,1)} \Rightarrow \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\Big|_{(2,3,1)}$

$= \left(8xy - 2y^2\right)\hat{i} + (4x^2 - 4xy)\hat{j} + 9z^2\hat{k}$

$= \left(8xy - 2y^2\right)\hat{i} + (4x^2 - 4xy)\hat{j} + 9z^2\hat{k}\Big|_{2,3,1} = \left[\left(8xy - 2y^2\right)\hat{i} + \left(4x^2 - 4xy\right)\hat{j} + 9z^2\hat{k}\right]\Big|_{(2,3,1)}$

$= \ 30\hat{i} - 8\hat{j} + 9\hat{k}$

Q2. The value of the first derivative of the function

$$f(x) = \frac{2}{\sqrt{3}} e^{-\sqrt{3}x^2|x|}$$

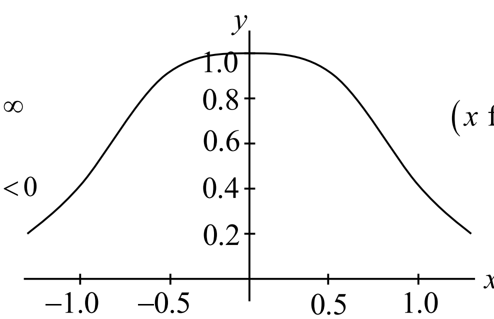
At $x = 0$ is $f'(0) = 0$

(a) 0 (b) 2 (c) $\frac{2}{\sqrt{3}}$ (d) Undefined

Topic-Mathematical Physics

Sub topic: Differentiation

Ans.: (a)

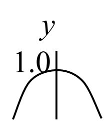


$\left(x \text{ from } -1.2 \text{ to } 1.2\right)$

Solution: $f(x) = \frac{2}{\sqrt{3}} e^{-\sqrt{3}x^2|x|} = \begin{cases} \frac{2}{\sqrt{3}} e^{-\sqrt{3}x^3}, & 0 < x < \infty \\ \frac{2}{\sqrt{3}} e^{\sqrt{3}x^3}, & -\infty < x < 0 \end{cases}$

$f'(x)\Big|_{x=0^+} = -\frac{2}{\sqrt{3}} \times 3x^2\sqrt{3}\, e^{-\sqrt{3}x^3}\Big|_{x=0} = 0$

$f'(x)\Big|_{x=0^-} = \frac{2}{\sqrt{3}} \times 3x^2\sqrt{3}\, e^{\sqrt{3}x^3}\Big|_{x=0} = 0$



Thus, $f'(0) = 0$

Q3. Consider a symmetric matrix

$$M = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \end{pmatrix}$$

An orthogonal matrix O which can diagonalize this matrix by an orthogonal transformation $O^T MO$ is given by O=

Topic-Mathematical Physics

Sub topic: Matrix

Ans. 3: (a)

Solution: $|M - \lambda I| = 0$

$$\begin{vmatrix} \frac{1}{3} - \lambda & 0 & \frac{1}{3} \\ 0 & 1-\lambda & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} - \lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[\left(\frac{1}{3} - \lambda\right)^2 - \frac{4}{9}\right] = 0$$

$$(1-\lambda)\left[\left(\frac{1}{3} - \lambda\right)^2 - \frac{4}{9}\right] = 0$$

$\therefore$ Eigen vector

for $\lambda = 1$

$$\begin{pmatrix} k_1 \\ y \\ k_1 \end{pmatrix}$$

for $k_1 = \frac{1}{\sqrt{2}}, y = 0, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

for $k_1 = 0, y = +1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda = -\frac{1}{3}$

$$\begin{pmatrix} 2/3 & 0 & 2/3 \\ 0 & 4/3 & 0 \\ 2/3 & 0 & 2/3 \end{pmatrix} = \begin{pmatrix} x \\ z \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

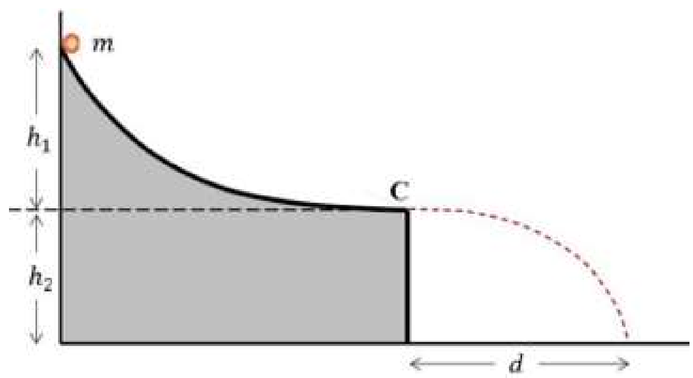
$x = -z$ and $y = 0$

Eigenvector for $\lambda = \frac{1}{3}$ is $\begin{pmatrix} k_1 \\ 0 \\ -k_1 \end{pmatrix}$

For $k_1 = \dfrac{1}{\sqrt{2}}$ it is $\begin{pmatrix} \dfrac{1}{\sqrt{2}} \\ 0 \\ -\dfrac{1}{\sqrt{2}} \end{pmatrix}$

Thus, $O = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$

Q4. A small body of mass m is released from rest at the top of a frictionless curved surface as shown in the figure, and permitted to slide down the curve. At the endpoint C, the tangent to the curve is horizontal. The mass then falls on the ground at a distance d as shown in the figure below when the experiment is carried out on the surface of the Earth. The heights h_1 and h_2 are also shown in the figure.



Suppose the same experiment is repeated on the surface of the Moon, where the acceleration due to gravity is $g' = \frac{g}{6}$, where g is the value on Earth. The corresponding distance d' at which the mass will fall on the ground in the Moon is

(a) d (b) $6d$

(c) $d\sqrt{\frac{h_1}{h_2}}$ (d) Dependent on the shape of the curve

Topic-Mechanics

Sub topic: Gravitation

Ans.: (a)

Solution: Conservation of energy:

$$\frac{1}{2}mv^2 = mgh_1$$

$$v_x = \sqrt{2gh_1}$$

$$h_2 = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h_2}{g}}$$

$$d' = v_x t = \sqrt{2gh_1}\sqrt{\frac{2h_2}{g}} = 2\sqrt{h_1h_2} = d$$

Q5. A particle is executing simple harmonic motion in a straight line. When the distance of the particle from the equilibrium position is x_1 and x_2, the corresponding values of its velocity are v_1 and v_2 respectively. The time period of oscillation is given by

(a) $2\pi\sqrt{\frac{x_2^{\,2}-x_1^{\,2}}{v_1^{\,2}-v_2^{\,2}}}$ (b) $2\pi\sqrt{\frac{x_2^{\,2}-x_1^{\,2}}{v_2^{\,2}-v_1^{\,2}}}$ (c) $2\pi\frac{x_2-x_1}{v_2-v_1}$ (d) $2\pi\frac{x_2-x_1}{v_1-v_2}$

Topic-Oscillation, Waves and Optics

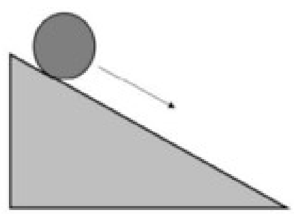
Sub topic: Oscillation

Ans. 5: (a)

Solution: $v_1 = \omega\sqrt{\left(A^2 - x_1^{\,2}\right)}$

$$v_2 = \omega\sqrt{\left(A^2 - x_2^{\,2}\right)},\ v_1^{\,2}-v_2^{\,2} = \omega^2(x_2^{\,2}-x_1^{\,2}) = \left(\frac{2\pi}{T}\right)^2(x_2^{\,2}-x_1^{\,2}) \Rightarrow T = 2\pi\sqrt{\frac{x_2^{\,2}-x_1^{\,2}}{v_1^{\,2}-v_2^{\,2}}}$$

Q6. A solid cylinder of uniform mass density rolls down a fixed inclined plane without slipping (see figure).



The fraction of the total kinetic energy of the cylinder associated with its rotation about its centre of mass is

Topic-Mechanics

Sub topic: Rotational Motion

Ans. 6: (a)

Solution: Total Kinetic energy is equals to = Translational K.E (T) + Rotational K.E (R)

$$k = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Cylinder Rolling without slipping

$$k=\frac{1}{2}mv^2+\frac{1}{2}\left(\frac{1}{2}mR^2\right)\frac{v^2}{R^2}=\frac{3}{4}mv^2$$

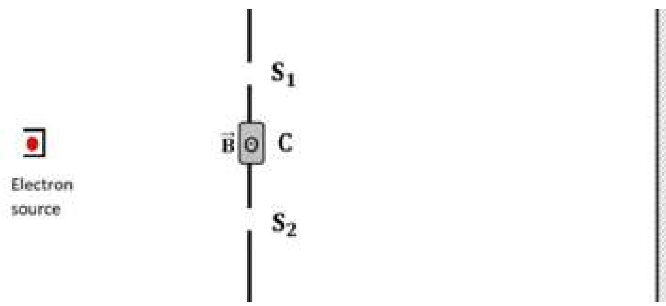
Rotational Kinetic energy

$$R=\frac{1}{2}\left(\frac{1}{2}mR^2\right)\frac{v^2}{R^2}=\frac{1}{4}mv^2$$

Now,

$$\frac{R}{R+K}=\frac{\frac{1}{4}mv^2}{\frac{3}{4}mv^2}=\frac{1}{3}$$

Q7. Consider an electron double slit experiment as shown in the figure below, with slits S_1 and S_2.



If now, within the shaded region marked C, a constant uniform magnetic field pointing outside the page is turned on, the fringe pattern

(a) will get shifted (b) will disappear

(c) will remain unchanged (d) will become dimmer

Topic-Oscillation, Waves and Optics

Sub topic: Waves

Ans. 7: (a)

Solution: $\vec{v}=v\hat{i}$, $B=B\hat{k}$

Force, $F=qv\times B=evB\hat{j}$. Thus, the pattern will shift in the j direction.

Q8. An atom of mass M at rest emits or absorbs a photon of frequency ν and recoils with a momentum p. The frequency of the internal transition of electronic levels is ν_0 without accounting for recoil. Assuming the process is nonrelativistic, the fractional differences between the photon frequency for emission and absorption $\frac{\nu-\nu_0}{\nu}$, respectively, are given by

(a) $\frac{-h\nu}{2Mc^2}$ (emission), $\frac{h\nu}{2Mc^2}$ (absorption) (b) $\frac{2h\nu}{Mc^2}$ (emission), $-\frac{2h\nu}{Mc^2}$ (absorption)

(c) $-\frac{2h\nu_0}{Mc^2}$ (emission), $\frac{2h\nu_0}{Mc^2}$ (absorption) (d) $\frac{2h\nu_0}{Mc^2}$ (emission), $-\frac{2h\nu_0}{Mc^2}$ (absorption)

Topic-Nuclear And Particle Physics

Sub topic: Particle Physics

Ans.: (a)

Solution: Absorption:

Energy conservation $h\nu = h\nu_0 + \frac{p^2}{2m}$

Momentum conservation $\frac{h\nu}{c} = p$

$$h\nu = h\nu_0 + \frac{(h\nu)^2}{2Mc^2} \Rightarrow \frac{\nu-\nu_0}{\nu} = \frac{-h\nu}{2Mc^2}$$

Emission: Energy conservation

$$h\nu_0 = h\nu + \frac{p^2}{2m}$$

Momentum conservation $\frac{h\nu}{c} = p$

$$h\nu_0 = h\nu + \frac{(h\nu)^2}{2Mc^2} \Rightarrow \frac{\nu-\nu_0}{\nu} = \frac{-h\nu}{2Mc^2}$$

$$\frac{\nu_0-\nu}{\nu} = \frac{h\nu}{2Mc^2}$$

Q9. Consider an electron with mass m_e, charge $-e$ and spin $\frac{1}{2}$, whose spin angular momentum operator is given by

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

This electron is placed in a magnetic field $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, where all three components (B_x, B_y, B_z) are nonvanishing.

At time $t = 0$, the electron is at rest in the $S_z = \frac{\hbar}{2}\sigma$ state. The earliest time when the state of the spin will be orthogonal to the initial state is

(a) Infinity, i.e., it will never be orthogonal

(b) $\frac{2m_e}{ge|B|}$

(c) $\frac{4m_e}{ge|B|}$

(d) Dependent on the direction of the magnetic field $\vec{B}$

Topic-Quantum Mechanics

Sub topic: Spin

Ans. 9: (b)

Solution: Hamiltonian of system is given $-\vec{\mu}.\vec{B} = -g\left(-\frac{e}{2m}\right)\vec{S}.\vec{B} = \frac{ge}{2m}|\vec{B}|\left(\vec{S}.\frac{\vec{B}}{|\vec{B}|}\right) = \frac{ge}{2m}\frac{\hbar}{2}\vec{\sigma}.\hat{n}$

Where $\vec{\sigma} = \sigma_x\hat{i} + \sigma_y\hat{j} + \sigma_z\hat{k}, \hat{n} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$

$\begin{pmatrix} \cos\theta & \exp{-i\phi}\sin\theta \\ \exp i\phi\sin\theta & -\cos\theta \end{pmatrix}$ the eigen vector corresponds to eigen value $E_0|B|$ and $-E_0|B|$ is given by $|\phi_1\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)\exp-\frac{i\phi}{2} \\ \sin\left(\frac{\theta}{2}\right)\exp\frac{i\phi}{2} \end{pmatrix}$ $|\phi_2\rangle = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)\exp-\frac{i\phi}{2} \\ \cos\left(\frac{\theta}{2}\right)\exp\frac{i\phi}{2} \end{pmatrix}$

The eigen value of Hamiltonian is $-E_0|B|, E_0|B|$ where $E_0 = \frac{ge\hbar}{4m}$ with orthonormal eigen vector like $|\phi_1\rangle$ and $|\phi_1\rangle$

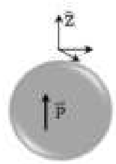
initially the state is $\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow |\psi\rangle = \cos\frac{\theta}{2}\exp\left(\frac{i\phi}{2}\right)|\phi_1\rangle - \sin\frac{\theta}{2}\exp\left(-\frac{i\phi}{2}\right)|\phi_2\rangle$

$$\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow |\psi\rangle = \cos\frac{\theta}{2}\exp\left(\frac{i\phi}{2}\right)|\phi_1\rangle \exp-\left(iEt/\hbar\right) - \sin\frac{\theta}{2}\exp\left(\frac{-i\phi}{2}\right)|\phi_2\rangle \exp-\left(-iEt/\hbar\right)$$

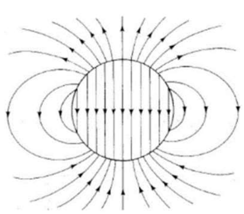
$$\langle\psi(0)|\psi(t)\rangle = \exp\left(-\frac{iEt}{\hbar}\right) + \exp\left(\frac{iEt}{\hbar}\right) = 0$$

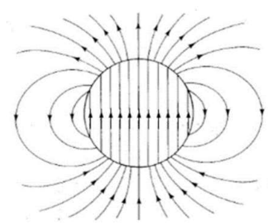
$$\cos\frac{Et}{\hbar} = \frac{\pi}{2} \Rightarrow t = \frac{\pi\hbar}{2E} = \frac{\pi\hbar}{2E_0|\vec{B}|} = \frac{\pi\hbar 4m}{2eg\hbar|\vec{B}|} = \frac{2m}{eg|\vec{B}|}$$

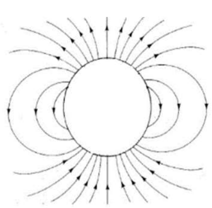
Q10. The electric field lines due to a uniformly polarized dielectric sphere (polarization along the positive Z-axis as shown in the figure)

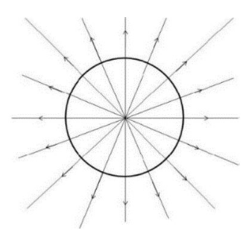


Will look like

(a) 

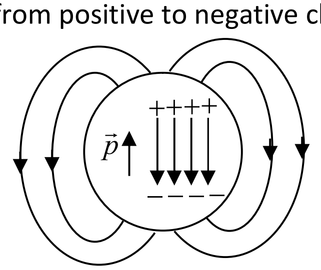
(b) 

(c) 

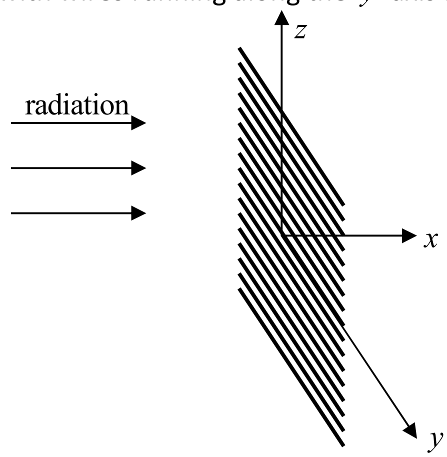
(d) 

Ans. (a)

Solution: The direction of polarization is from negative to positive charge. However, the direction of electric field is from positive to negative charge.



Q11. A beam of unpolarized microwave radiation is incident along the x-axis on a grid of metal wires in the yz-plane with wires running along the y-axis (see figure below).



If the width of each wire and the spacing between the adjacent wires is less than the wavelength of the microwave, the observation would be that

(a) The transmitted wave would be polarized along the z axis.

(b) The transmitted wave would be polarized along the y-axis.

(c) The transmitted wave would be unpolarized.

(d) No wave will pass through as the spacing is smaller than the wavelength

Topic-Oscillation, Waves and Optics

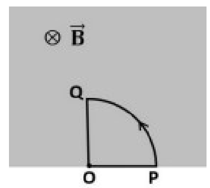
Sub topic: Oscillation

Ans.: (a)

Solution: The direction of polarization is perpendicular to the direction of propagation as well as perpendicular to the direction of chain or wire. This is the basic concept of polarization of EM waves.

Q12. Consider the following situation. A uniform magnetic field $\vec{B}$ pointing into the plane of the paper is present everywhere inside the rectangular region shown shaded in the adjoining figure. Outside the rectangular region, there is no magnetic field.

A closed loop of conducting wire is placed inside the rectangular region as shown in the figure at time $t = 0$. The loop is then rotated



counter clockwise with a uniform angular velocity ω about an axis perpendicular to the paper passing through the point O.

If the direction along PQOP is taken to be positive, then a correct graph for the EMF ε generated in the loop is

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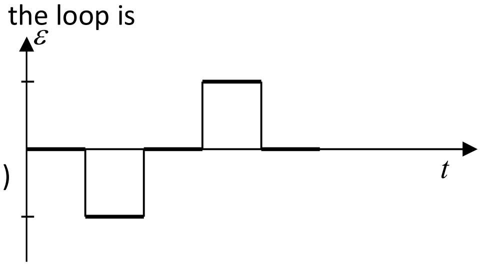

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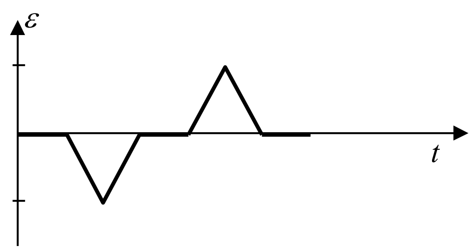

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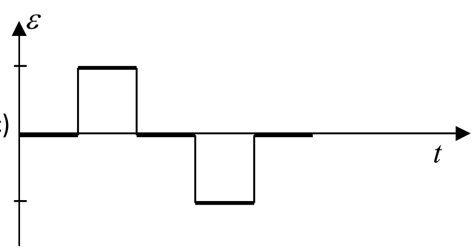

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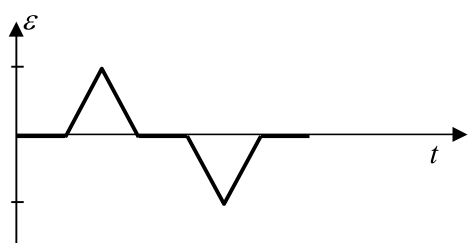


Topic-EMT

Sub topic: Electrodynamics

Ans.: (a)

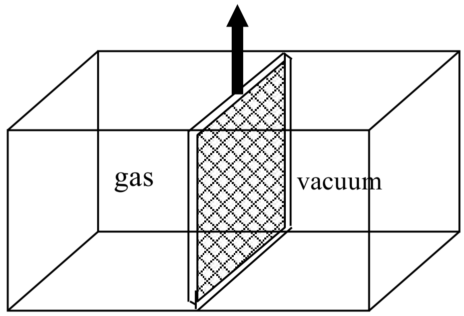
Solution: Induced emf, $\varepsilon = \dfrac{d\phi}{dt} = B\dfrac{dA}{dt}$

$\dfrac{dA}{dt}$ =Constant, B is uniform

Thus, ε is constant.

From Faraday's law we know that the system opposes, the change in flux. Since, flux through the loop decreases as rotates out of the B field, Emf will be induced in a direction which will increase the flux.

Q13. Consider a sealed but thermally conducting container of total volume V, which is in equilibrium with a thermal bath at temperature T. The container is divided into two equal chambers by a thin partition, which is thermally conducting but impermeable to particles. One of the chambers contains an ideal gas, while the other is a vacuum.



If the partition is removed suddenly and the ideal gas is allowed to expand and fill the entire container, then, once equilibrium has been reached, the entropy per molecule will increase by an amount

(a) $k_B \ln 2$ (b) $-k_B \ln 2$ (c) $2k_B$ (d) $\frac{k_B}{2}\ln 2$

Ans. (a)

Solution: $V_2 = 2V_1 \Rightarrow \frac{\Omega_1}{\Omega_2} = \left(\frac{V_1}{V_2}\right)^N \Rightarrow \frac{\Omega_1}{\Omega_2} = \left(\frac{1}{2}\right)^N$

$S_1 = k_B \ln \Omega_1$ & $S_2 = k_B \ln \Omega_2 = k_B \ln \Omega_1 + Nk_B \ln 2 \Rightarrow \frac{S_2 - S_1}{N} = \ln 2$

Q14. Five identical bosons are distributed in energy levels E_1 and E_2 with degeneracy of 2 and 3, respectively. Find the number of microstates if there are three bosons in the energy level E_1 and two bosons in the energy level E_2.

(a) 1024 (b) 120 (c) 24 (d) 6

Ans. (a)

Solution: No of ways $\pi_i \frac{(n+g-1)!}{n!(g-1)!} = \frac{4!}{3!}\frac{4!}{2!2!} = 24$

Q15. A two-level system with zero ground state energy is in equilibrium at a nonzero finite temperature. If α is defined as the ratio

$$\alpha = \frac{\langle E^2 \rangle}{\langle E \rangle^2}$$

Where $\langle E \rangle$ denotes the mean energy and $\langle E^2 \rangle$ denotes the mean squared energy, then

(a) $2 < \alpha < \infty$ (b) $1 < \alpha \le 2$ (c) $\frac{1}{2} < \alpha < 1$ (d) $0 < \alpha < \frac{1}{2}$

Topic-Thermodynamics and Statistical Mechanics

Sub topic: Distribution

Ans. (a)

Solution: $Z = 1 + e^{-\beta E_i}$; $\langle E \rangle = -\frac{1}{Z}\frac{dZ}{d\beta} = \frac{Ee^{-\beta E}}{1+e^{-\beta E}}$

$$\langle E^2 \rangle = \frac{1}{Z}\frac{d^2 Z}{d\beta^2} = \frac{E^2 e^{-\beta E}}{1+e^{-\beta E}}, \qquad a = \frac{\langle E^2 \rangle}{\langle E \rangle^2} = \frac{E^2 e^{-\beta E}}{1+e^{-\beta E}} \frac{\left(1+e^{-\beta E}\right)}{E^2 e^{-2\beta E}}$$

$$\alpha = 1 + e^{\beta E}; \qquad 2 < \alpha < \infty$$

Q16. The pulse train at the output of an XNOR gate with the three inputs

$$A = 00011011$$
$$B = 10100011$$
$$C = 00101110$$

will be

(a) 01101001 (b) 10010110 (c) 01010111 (d) 10101000

Topic-Electronics

Sub topic: Digital

Ans. (a)

Solution:

$$A \odot B = \bar{A}\bar{B} + AB$$

A	B	$A \odot B$	C		$(A \odot B) \odot C$
1	1	1	0		0
1	1	1	1		1
0	0	1	1		1
O	0	0	1	0	0
1	0	0	0		1
0	1	0	1		0
0	0	1	0		0
0	1	0	0		1

Correct answer is 01101001

Q17. In an experimental setup, positively charged particles are detected by a detector which requires a negative DC high voltage of -2000 V. Every time a charged particle is detected by the detector, it gives a transient pulse of height 10mV.

The data collection system used for the experiment needs to detect this pulse; however, it cannot operate at -2000 V. Which of the following circuits can be used to connect the data collection system to the detector to obtain these pulses?

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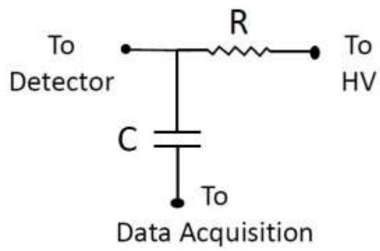

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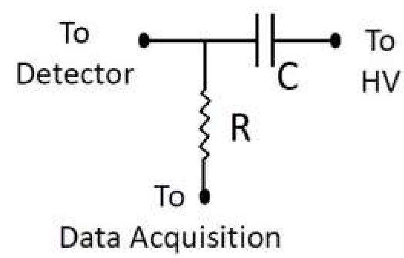

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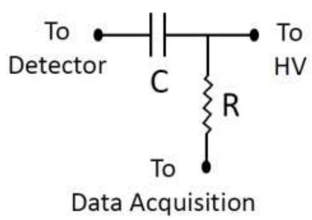

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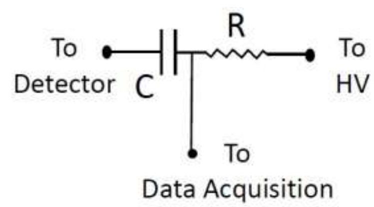


Topic-Electronics
Sub topic: ...

Ans. (a)

Solution: Data collection acquisition requires voltage less than -2000 V; it is possible to do that by connecting a capacitor parallel to the detector-voltage Source circuit. The capacitor will act as battery with voltage diff. less than - 2000 V

Q18. A faint star is known to emit light of a given frequency at an average rate of 10 photons per minute. An astronomer plans to measure this rate using a photon-counting detector. If she wants to measure the rate to an accuracy of 5%, approximately how long should be the exposure time?

(a) 40 minutes (b) 1 hour (c) 20 minutes (d) 10 minutes

Topic-Experimental Technique

Sub topic: Error analysis

Ans.: (a)

Solution: Assuming incoming photons follow Gaussian distribution then

Accuracy $= \frac{1}{\sqrt{N}}$

$\Rightarrow (0.05)^2 = \frac{1}{N} = \frac{1}{10\times 5},\qquad 10t = \frac{10000}{25} \Rightarrow t = 40 \text{ min}$

Q19. In a mercury vapour lamp, an electric arc passing through mercury vapour produces light. When the lamp is switched on, the arc is struck, and the liquid mercury starts evaporating as the temperature gradually increases.

In a certain experiment, a Michelson interferometer is set up with a mercury vapour lamp as the light source, and the lamp is switched on. Which of the following effects will be observed?

(a) Initially, fringes will appear with high contrast but low intensity, which will be reduced in contrast over the period of time as the light intensity builds up.

(b) Initially, no fringes will be observed, but then fringes will emerge with high contrast as the lamp heats up.

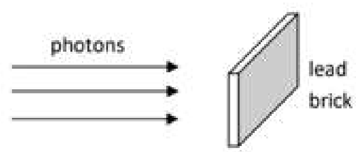
(c) No fringes will be observed as the source is incoherent and has many frequencies.

(d) High contrast fringes will appear as soon as the lamp is switched on and will remain thus so long as the lamp is on.

Ans.: (a)

Solution: Initially, fringes will appear with high contrast but low intensity, which will be reduced in contrast over the period of time as the light intensity builds up.

Q20. A beam of photons of $1MeV$ energy each is shot at a $10mm$ thick lead brick (see figure).

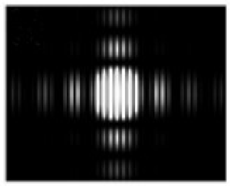


Given that the density of lead is $11.29g-cm^{-3}$, its atomic mass is 207.2 a.m.u., and also that the interaction cross-section for these photons with a lead atom is $10^{-23}cm^2$, the fraction of the incident photons that will cross the brick without losing any energy is

(a) 72% (b) 28% (c) 33% (d) 67%

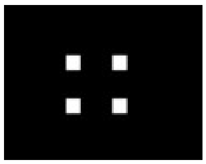
Ans. : (a)

Q21. The following Fraunhofer diffraction pattern was observed in an experiment.

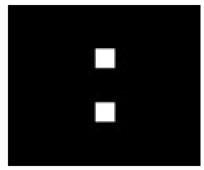


The aperture arrangement that would yield such a fringe pattern is

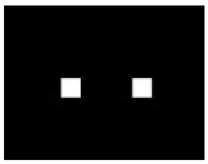
(a)



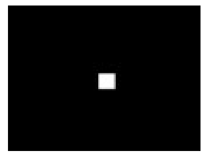
(b)



(c)



(d)



Ans. (a)

Topic-Oscillation, Waves and Optics

Sub topic: Diffraction

Q22. An electromagnetic wave is described by the following expression

$$\vec{E}(z,t) = E_0 \sin\, kz \left\{\hat{\imath}\cos\,\omega t + \hat{\jmath}\cos\left(\omega t - \frac{\pi}{2}\right)\right\}$$

Which of the following correctly describes this waveform?

(a) A left circular-polarised standing wave along the positive z-axis.

(b) A right circular-polarised standing wave along the positive z-axis.

(c) A left circular-polarised travelling wave along the positive z-axis.

(d) A right circular-polarised travelling wave along the positive z-axis.

Topic-Oscillation, Waves and Optics

Sub topic: Polarization

Ans. (a)

Solution: The equation of standing wave $u(x,t) = 2A\sin(kz)\cos(\omega t)$

Where the amplitude depends on the position and also function of t. The given equation

$$\vec{E}(z,t) = E_0 \sin\, kz \left\{\hat{\imath}\cos\,\omega t + \hat{\jmath}\cos\left(\omega t - \frac{\pi}{2}\right)\right\}$$

Clearly shows that there is a superposition of two standing waves having perpendicular state of polarization with equal amplitude for a given z at same time. Also, it is mirror of right CPL that is left circularly polarized for standing wave.

Q23. The minimum energy required to dissociate a hydrogen molecule H_2 into two atoms is $4.5eV$. If the electron affinity of the hydrogen atom is $0.75eV$, the minimum energy required to dissociate the hydrogen molecule into H^+ and H^- would be

(a) $17.35eV$ (b) $14.35eV$ (c) $18.85eV$ (d) $5.25eV$

Topic-Atomic, Molecular and Laser Physics

Sub topic: Atomic Physics

Ans. (a)

Solution: **Energy required is process 1 and 2**

1. $\mathrm{H_2} \rightarrow \mathrm{H} + \mathrm{H} \Rightarrow E_1 = 4 \cdot 5\mathrm{eV}$ 2. $H \rightarrow \mathrm{H}^+ + e^- \overset{0}{\rightarrow} E_2 = 13.6\mathrm{eV}$

Energy release in process 3

3. $\mathrm{H} + \mathrm{e}^- \rightarrow \mathrm{H}^-_{\rightarrow} \quad E_3 = 0.75\mathrm{eV}$

Thus, net energy required

$$E_1 + E_2 - E_3 = 17.35\,eV$$

Q24. The binding energy ε_b of a nuclide $^{Z}_{A}\mathrm{X}$ with atomic number Z and mass number A is given by the semi-empirical formula

$$\varepsilon_b = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} + a_A \frac{(A-2Z)^2}{A}$$

where the constant parameters and source of effect for each term are

Volume term	Surface term	Coulomb term	Asymmetry term
a_V	a_S	a_C	a_A
15.56MeV	17.8MeV	0.7MeV	23.29MeV

What is the mass difference between the two-mirror nuclei $_6C^{13}$ and $_7N^{13}$? It is known that both of them are spherical in shape and have a uniform charge distribution

(a) $2.62\,M\,eV$ (b) $3.40\,M\,eV$ (c) $0.78\,M\,eV$ (d) $1.84\,M\,eV$

Topic-Nuclear and Particle Physics

Sub topic: Shell Model

Ans.: (a)

Solution: The mass of a nucleus can be written as

$$M_n(Z,A) = Zm_p + Nm_p - \frac{B.E}{c^2}.,\ B.E. = a_v A - a_s A^{2/3} - \frac{a_c Z(Z-1)}{A^{1/3}} + a_a \frac{(A-2Z)^2}{A}$$

For carbon

$$M_n(6,13) = 6m_p + 7m_n - \frac{B.E}{c^2} ......(1)$$

For nitrogen

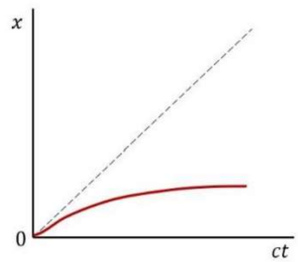
$$M_n(7,13) = 7m_p + 6m_n - \frac{B.E}{c^2} ....(2)$$

$$(2)-(1) \Rightarrow M_n(7,13) - M_n(6,13) = m_n - m_p + a_c \frac{(7^2 - 7 - 6^2 + 6)}{c^2}$$

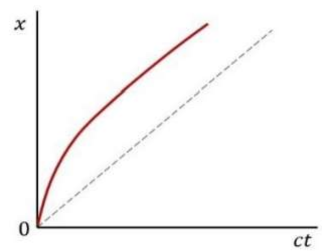
$$= 939.57 - 938.28 + 0.7 \frac{(7^2 - 7 - 6^2 + 6)}{(13)^{1/3} c^2} = 2.62$$

Q25. A relativistic particle, moving in one dimension x, starts from rest at $x = 0$ and is subjected to a uniform and constant force field along the positive x-direction. If the dashed line corresponds to $x = ct$, which of the following curves (red line) would best represent the position $x(t)$ of the particle?

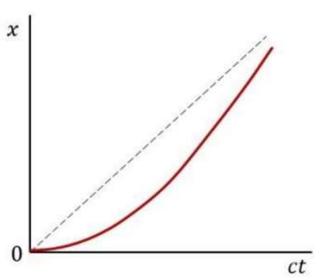
(a)



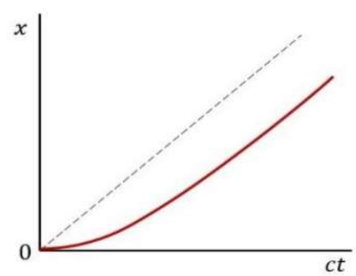
(b)



(c)



(d)



Topic: Classical Mechanics and STR

Sub topic: STR

Ans. : (a)

Solution: Trajectory of the particle should lie below the dashed bine and cannot cross the dashed line. This is because dashed line con be thought of as trajectory followed by a massless particle moving with 'c'

Section B

Q1. A random positive variable x follows an exponential distribution

$$p(x) \propto e^{-\lambda x}$$

With $\lambda > 0$. The probability of observing an event $x > 3\langle x \rangle$,where $\langle x \rangle$ represents the average value of x is

(a) $\frac{1}{e^3}$ (b) $\frac{1}{e^4}$ (c) $\frac{1}{e}$ (d) $\frac{1}{e^2}$

Topic-Mathematical Physics

Sub topic: Probability

Ans.: (a)

Solution: $p(x) = k\, e^{-\lambda x}$

Total probability is one, which implies

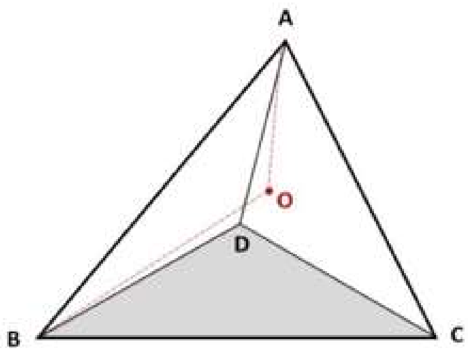
$$k\int_0^\infty e^{-\lambda x} dx = 1 \Rightarrow \frac{k}{\lambda} = 1 \Rightarrow k = \lambda$$

$$\langle x \rangle = \lambda \int_0^\infty x e^{-\lambda x} dx = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

The probability of observing an event $x > 3\langle x \rangle$ is

$$p = \lambda \int_{3/\lambda}^\infty e^{-\lambda x} dx = \frac{\lambda}{-\lambda} e^{-\lambda x}\Big|_{3/\lambda}^\infty = 1/e^3$$

Q2. Consider a regular tetrahedron $ABCD$, as shown in the figure below. Let the point O be its centre.



The value of the angle AOB must be

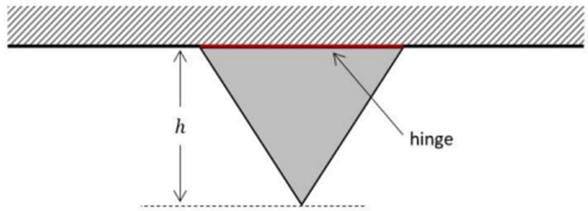
(a) $\cos^{-1}(-1/3)$ (b) $\cos^{-1}(-4/5)$ (c) $\cos^{-1}(-\sqrt{4/5})$ (d) $2\pi/3$

Topic-Mathematical Physics

Sub topic: Geometry

Ans. (a)

Q3. A thin equilateral triangular plate of uniform mass density is attached to a fixed horizontal support along one of its sides through a frictionless hinge, as shown in the figure below. The vertical distance between the rod and the lower tip of the plate is h



If the pointed tip of the plate is displaced (out of the plane of the paper) so that its plane forms a small angle with the vertical plane passing through the rod, the angular frequency ω of the resultant motion is $\omega =$

(a) $\sqrt{\frac{2g}{h}}$ (b) $\sqrt{\frac{2\sqrt{3}g}{h}}$ (c) $\sqrt{\frac{2g}{\sqrt{3}h}}$ (d) $\sqrt{\frac{\sqrt{3}g}{2h}}$

Topic-Classical Mechanics

Sub topic: Moment of inertia

Ans.: (a)

Let us consider the side of triangle is a . Then we can write $a=\frac{2h}{\sqrt{3}}$

The hight of the of centroid from each base is $\frac{a}{2\sqrt{3}}=\frac{h}{3}$

The M.I. about the centroid is

$$I_z=\frac{M}{3}\left(\frac{a}{2\sqrt{3}}\right)^2 3=\frac{Ma^2}{12}$$

The moment of inertia about centre of mass perpendicular to plane is $I_z=\frac{Ma^2}{12}$ where $a=\frac{2h}{\sqrt{3}}$

The distance of centre of mass from any base is $\frac{a}{2\sqrt{3}}=\frac{h}{3}$

From perpendicular axis theorem $I_z=I_x+I_y$ from symmetry of equilateral triangle

$$I_x=I_y\Rightarrow I_x=\frac{Ma^2}{24}=\frac{Mh^2}{18}$$

The moment of inertia of plate about axis passes through hinge we can use parallel axis theorem

$$I=\frac{Mh^2}{18}+\frac{Mh^2}{9}\Rightarrow\frac{3Mh^2}{18}=\frac{Mh^2}{6}$$

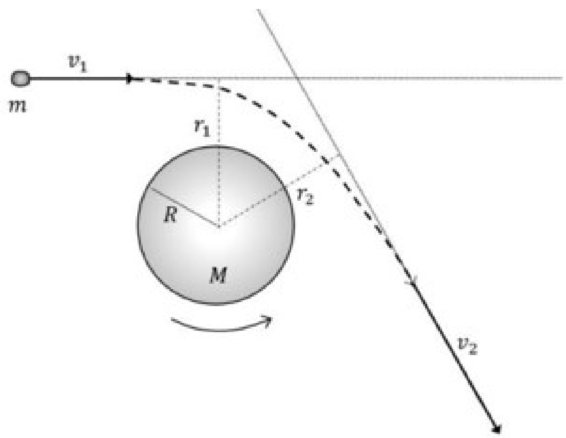
The plate is executing pure rotation about hinge so

$$E=\frac{1}{2}I\dot{\theta}^2-Mg\frac{h}{3}\cos\theta\Rightarrow E=\frac{1}{2}\frac{Mh^2}{6}\dot{\theta}^2-\frac{Mgh}{3}\cos\theta$$

Total energy of system is conserve $\frac{dE}{dt}=\frac{Mh^2\dot{\theta}\ddot{\theta}}{6}+\frac{Mgh}{3}\sin\theta\dot{\theta}=0\Rightarrow\frac{Mh^2\ddot{\theta}}{6}+\frac{Mgh}{3}\theta=0$

$$\ddot{\theta}+\frac{2g}{h}\theta=0\Rightarrow\omega=\sqrt{\frac{2g}{h}}$$

Q4. A spherical planet of mass M, radius R and uniform density is rotating anticlockwise about an axis passing through its centre, which, in the figure below, is normal to the plane of the paper. The duration of a 'day' on this planet is T



A small asteroid of mass m approaches the above planet from far away with a uniform speed v_1 along a straight line at a perpendicular distance r_1 from the centre of the planet (see figure). This path gets distorted by the gravitational field of the planet, and the asteroid leaves with a final uniform speed v_2 along a straight line at a perpendicular distance r_2 from the centre of the planet.

It is observed that after the passage of the asteroid, the length of the day on the planet has changed by $\delta T=$

(a) $\frac{5T^2}{4\pi}\frac{m(v_2r_2-v_1r_1)}{MR^2}$

(c) $\frac{4\pi}{5T^2}\frac{MR^2}{m(v_2r_2-v_1r_1)}$

(c) $\frac{5}{4\pi}\frac{MR^2}{m(v_2r_2-v_1r_1)}$

(d) 0

Topic-Classical Mechanics

Sub topic: Conservation of Angular momentum

Ans. : (a)

Solution: In this problem angular momentum of Asteroid is not conserve so we know rate of change in angular momentum is equal to external torque. This torque is produced by change in angular velocity of planet due to interaction between planet and asteroid.

$$\frac{dL}{dt} = I\alpha = \frac{2}{5}MR^2\frac{d\omega}{dt} \Rightarrow dL = \frac{2}{5}MR^2 d\omega \Rightarrow$$

$$|dL| = \frac{2}{5}Mv^2\frac{2\pi}{T^2}|dT| \Rightarrow m(v_2r_2 - v_1r_1) = \frac{2}{5}MR^2\frac{2\pi}{T^2}|dT| \Rightarrow dT = \frac{5mT^2(v_2r_2 - v_1r_1)}{4\pi MR^2}$$

Q5. A particle is in the ground state of a one-dimensional box

$$-\frac{L}{2} \le x \le \frac{L}{2}$$

The uncertainty product $\Delta x \Delta p$ for this state satisfies

(a) $\frac{\hbar}{2} \le x \le \hbar$ (b) $\hbar \le x \le 3\frac{\hbar}{2}$ (c) $3\frac{\hbar}{2} \le x \le 2\hbar$ (d) $\Delta x \Delta p = \frac{\hbar}{2}$

Topic-Quantum Mechanics

Sub topic: Heisenberg Uncertainty Principle

Ans.: (a)

Solution: For a given system $\langle x \rangle = 0, \langle x^2 \rangle = \frac{L^2}{12} - \frac{L^2}{2\pi^2}$ $\langle p_x \rangle = 0 \Rightarrow \langle p_x^2 \rangle = \frac{\pi^2\hbar^2}{L^2}$

$$\Delta x.\Delta p_x = L\sqrt{\frac{1}{12} - \frac{1}{2\pi^2}} \times \frac{\pi\hbar}{L} = 0.57\hbar$$

Q6. Consider a diatomic molecule with two atoms of masses $m_1 = 1\,amu$ and $m_2 = 1\,amu$ which are separated by a distance r and bound by an effective interaction potential of the form

$$V(r) = \varepsilon\left(\frac{a^4}{4r^4} - \frac{b^2}{2r^2}\right)$$

Where $\varepsilon = 4\times10^{-18}\,J$, $a = b = 1\,A^\circ$ and $1\,amu = 1.6\times10^{-27}\,kg$

Making a small oscillations approximation, the transition frequency corresponding to the vibrational spectra of the molecule is approximately

(a) $1.2\times10^{14}\ Hz$ (b) $0.4\times10^{14}\ Hz$ (c) $7.5\times10^{14}\ Hz$ (d) $3.6\times10^{14}\ Hz$

Topic-Classical Mechanics

Sub topic: Small Oscillation

Ans. (a)

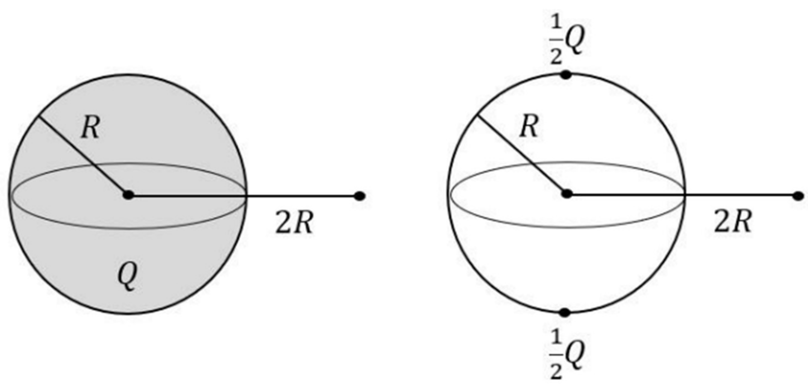
Solution: $\frac{dv}{dr} = 0 \Rightarrow r = 1\,A^o$

$\frac{d^2v}{dr^2} = k \Rightarrow k = \varepsilon r^{-2} = \varepsilon(1A^o)^{-2}$

And $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{2} amu = \frac{1}{2} \times 1.6 \times 10^{-27}\ kg$

$\omega = \sqrt{\frac{\frac{d^2v}{dr^2}}{\mu}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{\left.\frac{\partial^2 v}{\partial r^2}\right|_{r=1A^0}}{\mu}} = 1.2 \times 10^{14}\ Hz$

Q7. Consider a solid sphere of radius R with a total charge Q distributed uniformly throughout its volume (see figure, left). The electric field measured at a distance $x = 2R$ from the centre of the sphere along the equatorial plane is found to be E_1



Next, the same charge is distributed differently, such that $\frac{Q}{2}$ is concentrated at the north pole, and the remaining $\frac{Q}{2}$ is concentrated at the south pole (see figure, right). The electric field is measured again at the same point on the equatorial plane and found to be E_2

The value of $\frac{E_2}{E_1}$

(a) $\frac{8}{5\sqrt{5}}$ (b) 1 (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{4}{5}$

Topic-EMT

Sub topic: Electrostatic

Ans. (a)

Solution: $E_1 = \frac{Q}{64\pi\varepsilon_0 R^2}$

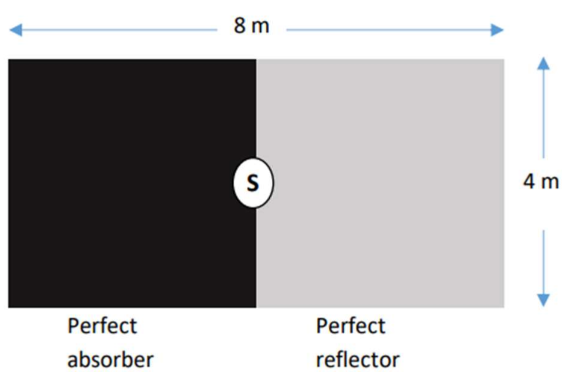
In second case $E_2 = \dfrac{2\dfrac{Q}{2}\cos\theta}{4\pi\varepsilon_0 \times 5R^2} = \dfrac{Q}{2\pi\varepsilon_0 \times 5\sqrt{5}R^2}$

Now $\cos(\theta) = \dfrac{2}{\sqrt{5}}$

Thus, $E_2 = \dfrac{2\dfrac{Q}{2}\dfrac{2}{\sqrt{5}}}{4\pi\varepsilon_0 \times 5R^2} = \dfrac{Q}{2\pi\varepsilon_0 \times 5\sqrt{5}R^2}$

$$\frac{E_2}{E_1} = \frac{8}{5\sqrt{5}}$$

Q8. A small satellite S has a rectangular solar sail of dimensions $8m \times 4m$ which propels the satellite upon receiving sunlight. One half of the sail is a perfect reflector, while the other half is a perfect absorber, as shown in the figure.



Assuming uniform sunlight incident normally on the sail with an intensity $1370Wm^{-2}$ and ignoring the satellite's shadowing effects, the instantaneous torque experienced by the satellite is

(a) $1.46 \times 10^{-4}\ NM$ (b) $2.92 \times 10^{-4}\ NM$ (c) $0.73 \times 10^{-4}\ NM$ (d) $2 \times 10^{-4}\ NM$

Topic-Classical Mechanics

Sub topic: Torque

Ans.: (a)

Solution: The intensity can be written as

$$I = \frac{P}{A} = P = IA \Rightarrow F.c = IA \Rightarrow$$

So, the force can be written as

$$F = \frac{IA}{c} \Rightarrow \vec{\tau} = \vec{r} \times \vec{F} = \frac{IA}{c} = 2\frac{1370 \times 4 \times 4}{3 \times 10^8} = 1.46 \times 10^{-4}\, N - m$$

Q9. The equilibrium temperature (T_0) on the surface of a planet results from the balance between the energy received from their host star and the energy they emit back into space. In the case of the Earth, $T_0 = 300\,k$ and the orbit is almost circular. We may safely assume that planets absorb and emit radiation like perfect blackbodies

Now consider an exoplanet of the same size as the Earth, which orbits a fainter star having a power output only 25% of that of the Sun, in an almost-circular orbit of radius 25% that of the Earth-Sun distance. The equilibrium temperature T_0' on the surface of this exoplanet will be about

(a) $424\,k$ (b) $212\,k$ (c) $300\,k$ (d) $600\,k$

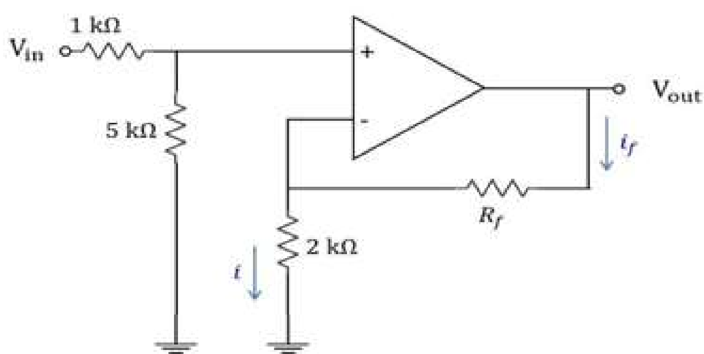
Topic-Quantum Mechanics

Sub topic: Black Body Radiation

Ans. (a)

Solution: $I = \frac{P}{A} = \frac{16P_s}{4A_s} = \frac{4P_s}{A_s} = 4 \times I_E \Rightarrow 4 \times 300^4 = T^4 \Rightarrow T = 300 \times 1.41 = 424\,k$

Q10. At what value of R_f will the ideal op-amp shown in the figure provide a gain of 6?



(a) $12.4\,k\Omega$ (b) $19.5\,k\Omega$ (c) $22.5\,k\Omega$ (d) $14.4\,k\Omega$

Topic-Electronics

Sub topic: Op Amp

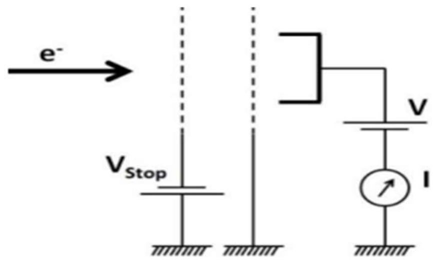
Ans. (a)

Solution: $V_s = v_{in} \times \frac{5}{5+1} = v_{in} \times \frac{5}{6}$

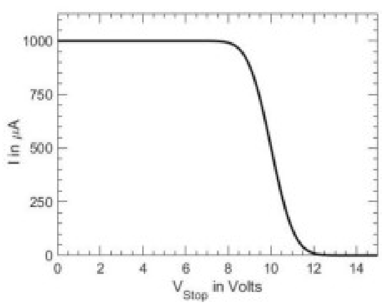
Non inverting amplifier.

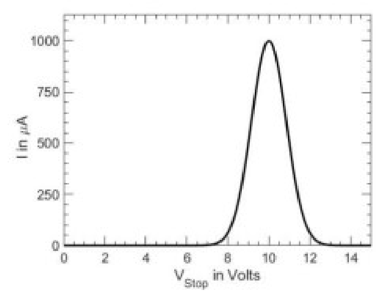
So, $V_{out} = (1 + \frac{R_f}{2}) v_{in} \times \frac{5}{6}$, $\frac{V_{out}}{v_{in}} = (1 + \frac{R_f}{2}) \frac{5}{6} = 6 \Rightarrow R_f = 2\left(\frac{36}{5} - 1\right) = 12.4$

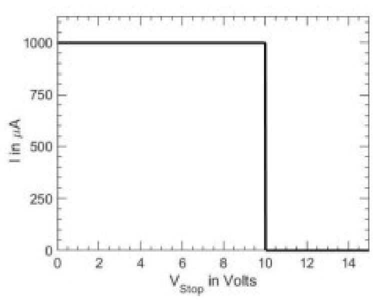
Q11. A well-collimated constant-current electron beam of Gaussian energy distribution centred at $10eV$ with FWHM of $2eV$ is detected using a metal cup connected to an ammeter, as shown in the figure below. The entire apparatus is kept in vacuum.

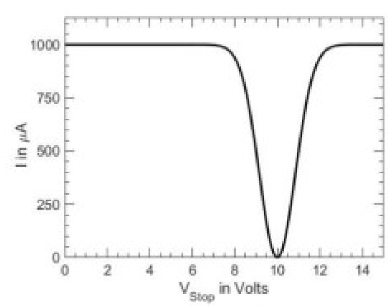


To measure the energy width of the electron beam, a grid is introduced with a voltage source V_{stop} connected to it, as shown in the figure. The current measured in the cup is plotted as a function of the value of V_{stop}. The graph of the current I vs V_{stop} would

(a)


(b)


(c)


(d)


Ans. (a)

Q12. A diffraction grating spectrograph is used to resolve the two sodium D lines (589 nm and 589.6 nm) in the first order of diffraction. If the width of the grating is 2cm and the focal length of the spectrograph camera is 20 cm, what the linear separation at the focal plane of the two D lines will be about

(a) $6\,\mu m$ (b) $6\,mm$ (c) $60\,\mu m$ (d) $60\ nm$

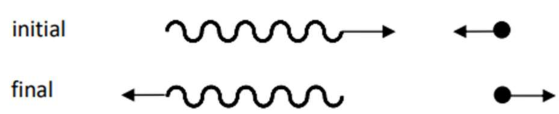
Topic-Oscillation, Waves and Optics

Sub topic: Diffraction

Ans. : (a)

Solution: $\Delta y = \frac{Dm\Delta\lambda}{d} = \frac{Dm\Delta\lambda}{\lambda} = \frac{20\times 0.6}{589} \approx 6\mu m$

Q13. A photon of frequency ν_i collides "head on" with an electron of mass m having speed u_i and the photon scatters off in a direction exactly opposite to its initial momentum (see figure).



It is found that the frequency of the scattered photon is the same as that of the incident photon. Which of the following conditions must hold for this to happen?

(a) The magnitude of the initial momentum of the electron is $p_i^e = h\nu_i / c$

(b) The initial energy of the electron is $E_i^e = h\nu_i$

(c) The magnitude of the initial momentum of the electron is $p_i^e = 2h\nu_i / c$

(d) The initial energy of the electron is $E_i^e = 2h\nu_i$

Topic-Nuclear and Particle Physics

Sub topic: Particle Physics

Ans.: (a)

Solution: The given interaction

$\gamma + e^- \to \gamma + e^-$

The linear momentum should be conserved in any reaction.

Let us assume the frequency and momentum of incident photon are ν_i and p_i respectively.

We can write $p_i = \frac{h\nu_i}{c}$

According to the given statement, frequency and momentum of scattered photon are ν_i and $-p_i$ respectively.

Let assume the linear momentum of incident electron is $-p_e$

Then, from linear momentum of scattered electron is p_e

Thus, from liner momentum conservation

$p_i - p_e = -p_i + p_e$

$p_e = p_i = \frac{h\nu_i}{c}$

Q14. The number of hyperfine states found in the He^3 atom for the electronic configuration

$$1s^1 2s^0 2p^1$$

would be

(a) 7 (b) 2 (c) 4 (d) 1

Topic-Atomic, Molecular and Laser Physics

Sub topic: Hyperfine structure

Ans. : (a)

Solution: $l_1 = 0,\ l_2 = 1,\ \Rightarrow L = 1$

$s_1 = \frac{1}{2}, s_2 = \frac{1}{2} \Rightarrow S = 1,0$ & $S = 0, L = 1 \Rightarrow J = 1,$

$I = \frac{1}{2}\ For\ He^3 \Rightarrow F = \frac{3}{2}, \frac{1}{2}$

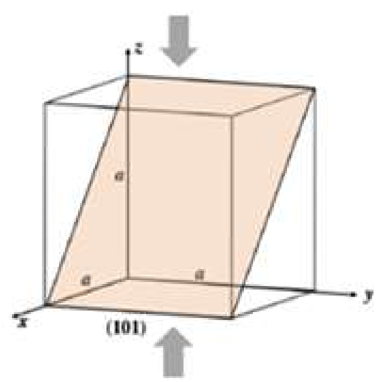
$S = 1, L = 1, J = 2,1,0$, $S = 1, L = 1, J = 2, I = \frac{1}{2}, F = \frac{5}{2}, \frac{3}{2}$

$S = 1, L = 1, J = 1, I = \frac{1}{2}, F = \frac{3}{2}, \frac{1}{2}$, $S = 1, L = 1, J = 0, I = \frac{1}{2}, F = \frac{1}{2}$

To total seven F values are obtained for different states. So, number of hyperfine splitting is seven.

Q15. An X-ray of wavelength λ, when incident on the (101) plane of a cubic lattice with lattice constant a produces a first-order Bragg's reflection at $\theta = 30^0$ (θ) is measured from the lattice plane).

Suppose this cubic lattice is compressed along the z axis such that its lattice parameters along the x and y axes remain the same while that along the z axis becomes $\frac{a}{\sqrt{3}}$ (see figure).



The first-order reflection for the (101) plane of the compressed lattice occurs at:

(a) $\theta = 45^0$ (b) $\theta = 15^0$ (c) $\theta = 30^0$ (d) $\theta = 60^0$

Topic-Solid state Physics

Sub topic: XRD

Ans.: (a)

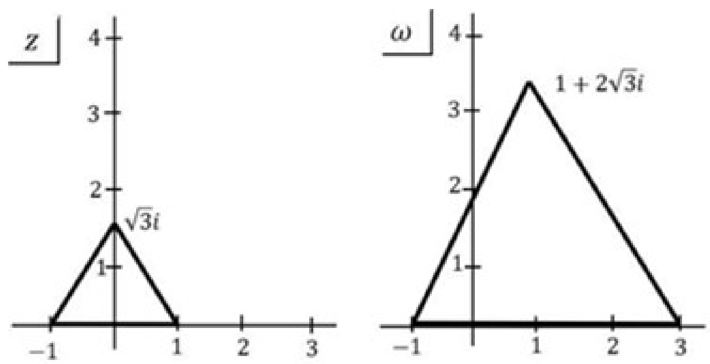
Solution: $2d\sin(\theta)=\lambda \Rightarrow 2d\sin(30)=\lambda \Rightarrow \lambda=\frac{a}{\sqrt{2}}\ \ [a=b=c]$

$a=b,\ c=\frac{a}{\sqrt{3}}$ & $2d'\sin(\theta')=\lambda$

$$\sin(\theta')=\frac{\frac{a}{\sqrt{2}}}{2\times\frac{a}{2}}=\frac{1}{\sqrt{2}}\Rightarrow \theta'=45^0$$

Section C

Q1. A complex analytic function $\omega=f(z)$ transforms an equilateral triangle in the complex z-plane to another equilateral triangle in the complex ω-plane as shown in the figure.



Which of the options below can be $f(z)$?

(a) $f(z)=e\frac{5\pi i}{6}z+2i\sqrt{3}$

(b) $f(z)=2e\frac{2\pi i}{3}z+2+i\sqrt{3}$

(c) $f(z)=2e\frac{2\pi i}{3}z+i\sqrt{3}$

(d) $f(z)=2z+i\sqrt{3}$

Topic-Mathematical Physics

Sub topic: Complex algebra

Ans.: (a)

Solution: Only this $f(z)=e\frac{5\pi i}{6}z+2i\sqrt{3}$ does not map the coordinates of triangle in z-plane to triangle in ω-plane

Q2. Calculate the integral

$$\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x^2+1)}$$

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\pi\sqrt{2}$ (c) 2π (d) π

Topic-Mathematical Mechanics

Sub topic: Complex analysis

Ans. : (a)

Solution: $x = \tan(\theta) \Rightarrow dx = \sec^2(\theta)d\theta$

$$\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x^2+1)} = \int_0^{\pi/2} \frac{\sec^2\theta d\theta}{\sec^2\theta\sqrt{\tan(\theta)}} = \int_0^{\pi/2} \frac{\cos\theta^{1/2}\sin\theta^{-1/2} \quad d\theta}{1} = \frac{\overline{)3/2} \times \overline{)1/2}}{2} = \frac{\pi}{4}$$

Q3. Consider the Hamiltonian for a one-dimensional classical oscillator

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$

A canonical transformation to variables (P, Q) via the generating function

$$F = \frac{m\omega q^2}{2}\cot\, Q$$

leads to which of the following Hamiltonians in the new coordinates?

Topic-Classical Mechanics

Sub topic: Canonical Transformation

Ans. : (a)

Solution: $F_1 = \frac{m\omega^2 q^2}{2}\cot Q$

$$\frac{\partial F_1}{\partial Q} = -P \Rightarrow \frac{m\omega^2 q^2}{2}\left(-\cos ec^2 Q\right) = -P \Rightarrow q = \sqrt{\frac{2P}{m\omega^2}}\sin Q$$

$\frac{\partial F_1}{\partial q} = p \Rightarrow m\omega^2 q\cot Q = p$ put the value of $q = \sqrt{\frac{2P}{m\omega^2}}\sin Q$

One will get $p = \sqrt{m\omega P}\cos Q$

$K = H + \frac{\partial F_1}{\partial t}$ where $\frac{\partial F_1}{\partial t} = 0$ so

$$K = H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

Now we will put the value of $q=\sqrt{\frac{2P}{m\omega^2}}\sin Q$ and $p=\sqrt{m\omega P}\cos Q$ we will get $K=\omega P$

OR

$$H=\frac{1}{2m}\left(p^2+m^2\omega^2q^2\right)$$

$$F=\frac{m\omega q^2}{2}\cot Q$$

$$p=\frac{\partial F}{\partial q}=m\omega q\cot(Q)\,,\quad p=\frac{\partial F}{\partial q}=m\omega q\cot(Q)\,,\quad P=-\frac{\partial F}{\partial Q}=\frac{m\omega q^2}{2}\operatorname{co}esc^2(Q)$$

$$H=\frac{\left(m\omega q\right)^2}{2m}\cos ec^2(Q)=\frac{m\omega^2q^2}{2}\cos ec^2(Q)=\omega P$$

Q4. A simple pendulum is oscillating freely in the vertical plane. If the string is shortened very slowly to half its length, the angular amplitude $\theta_{\max}$ will change by a factor

(a) $2^{3/4}$ (b) $\sqrt{2}$ (c) 2 (d) Does not change.

Topic-Oscillation, Waves and Optics

Sub topic: SHM

Ans.: (b)

Solution: Energy should be conserved

$$mgl(1-\cos\theta_{\max})=mg\frac{l}{2}(1-\cos\theta'_{\max})\Rightarrow 2\sin^2(\frac{\theta_{\max}}{2})=\sin^2(\frac{\theta'_{\max}}{2})$$

$$\Rightarrow 2\sin^2(\frac{\theta_{\max}}{2})=\sin^2(\frac{\theta'_{\max}}{2})\Rightarrow 2^{1/2}\sin(\frac{\theta_{\max}}{2})=\sin(\frac{\theta'_{\max}}{2})\Rightarrow 2^{1/2}(\frac{\theta_{\max}}{2})=(\frac{\theta'_{\max}}{2})$$

$$2^{1/2}\theta_{\max}=\theta'_{\max}\Rightarrow\theta'_{\max}=2^{1/2}\theta_{\max}$$

We are getting option (b).

Q5. Consider a particle of mass m in a quartic potential

$$H=\frac{p^2}{2m}+ax^4$$

If we take a variational wavefunction

$$\varphi(x,\lambda)=e^{-\lambda x^2}$$

with $\lambda>0$ and try to estimate the ground state energy, the value of λ should be chosen as

[You may use the integral

$$\int_{-\infty}^{\infty}(A+Bx^2+Cx^4)dx = A\sqrt{\frac{\pi}{\lambda}}+\frac{B}{2}\sqrt{\frac{\pi}{\lambda^3}}+\frac{3C}{4}\sqrt{\frac{\pi}{\lambda^5}}$$

Where A, B,C and $\lambda>0$ are all constants.]

(a) $\left(\frac{3ma}{4\hbar^2}\right)^{1/3}$ (b) $\left(\frac{5ma}{3\pi^2\hbar^2}\right)^{1/3}$ (c) $\left(\frac{15ma}{8\hbar^2}\right)^{1/3}$ (d) $\left(\frac{ma}{2\pi\hbar^2}\right)^{1/3}$

Topic-Quantum Mechanics

Sub topic: Variational Principle

Ans.: (a)

Solution: $\psi = Ae^{-\lambda x^2}$

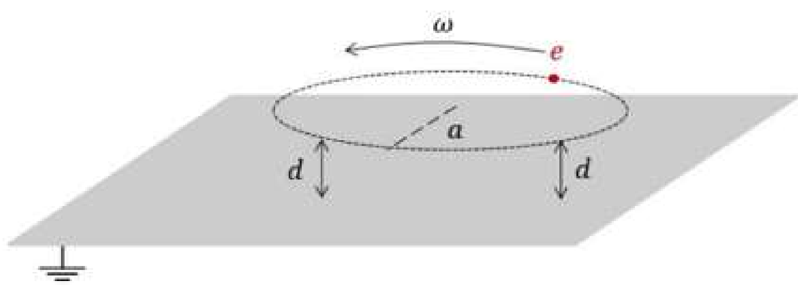
$$\int_{-\infty}^{\infty}|\psi|^2dx=1\Rightarrow\int_{-\infty}^{\infty}A^2e^{-2\lambda x^2}dx=1\Rightarrow A^2\sqrt{\frac{\pi}{2\lambda}}=1\Rightarrow A^2=\sqrt{\frac{2\lambda}{\pi}}$$

$$\langle H\rangle=\frac{2\lambda\hbar^2}{2m}-\frac{4\lambda^2\hbar^2}{2m}\frac{A^2}{2}\sqrt{\frac{\pi}{8\lambda^3}}+\frac{3}{4}A^2\alpha\sqrt{\frac{\pi}{32\lambda^5}}$$

$$\langle H\rangle=\frac{2\lambda\hbar^2}{2m}-\frac{4\lambda^2\hbar^2}{2m}\frac{A^2}{2}\sqrt{\frac{\pi}{8\lambda^3}}+\frac{3}{4}A^2\alpha\sqrt{\frac{\pi}{32\lambda^5}},\ =\frac{2\lambda\hbar^2}{2m}-\frac{4\lambda^2\hbar^2}{2m}\frac{\sqrt{\frac{2\lambda}{\pi}}}{2}\sqrt{\frac{\pi}{8\lambda^3}}+\frac{3}{4}\sqrt{\frac{2\lambda}{\pi}}\alpha\sqrt{\frac{\pi}{32\lambda^5}}$$

$$\frac{d}{d\lambda}\langle H\rangle=0\Rightarrow\frac{\hbar^2}{m}-\frac{\hbar^2}{2m}+\frac{3}{16}\alpha\frac{-2}{\lambda^3}=0,\ \lambda=\left(\frac{3ma}{4\hbar^2}\right)^{1/3}$$

Q6. A charge e is moving with an angular frequency ω along a circle of radius a always keeping a small distance $d\ (d<<a)$ from a grounded infinite conducting plane.



The leading dependence of the radiated power $P(\omega)$ at a distance $r(r>>a)$ will be

(a) $P(\omega)\,\alpha\,\omega^4$ (b) $P(\omega)\,\alpha\,\omega^6$ (c) $P(\omega)\,\alpha\,\omega^3$ (d) $P(\omega)\,\alpha\,\omega^2$

Topic-EMT

Sub topic: Radiation

Ans.: (a)

Solution: $p = qx(t);\ x(t) = A\cos(\omega t)$

$$\dot{p} = -qA\omega\sin(\omega t)$$

$$\ddot{p} = -qA\omega^2\cos(\omega t) \Rightarrow \left\langle \ddot{p} \right\rangle^2 = q^2A^2\omega^4$$

Q7. For an electromagnetic wave propagating through a rectangular waveguide, the electric and magnetic fields

(a) are never perpendicular to each other

(b) are always perpendicular to each other

(c) are perpendicular to each other only in the TE mode

(d) are perpendicular to each other only in the TM mode

Topic-EMT

Sub topic: Wave guide

Ans.: (a)

Solution: The electric and magnetic field in a rectangular waveguide is never perpendicular to each other.

Q8. Consider a fermionic system with a Hamiltonian

$$\hat{H} = \begin{bmatrix} 0 & E_0 & 0 \\ E_0 & 0 & 2E_0 \\ 0 & 2E_0 & 0 \end{bmatrix}$$

Consider the grand canonical ensemble of this system at temperature T and zero chemical potential, where k_B is the Boltzmann constant. The grand canonical partition function of the system is given by

(a) $4 + \cosh\left(\sqrt{5}\frac{E_0}{k_BT}\right)$

(b) $\cosh\left(\sqrt{5}\frac{E_0}{k_BT}\right)$

(c) $\frac{1}{4+\cosh\left(\sqrt{5}\frac{E_0}{k_BT}\right)}$

(d) $\operatorname{sech}\left(\sqrt{5}\frac{E_0}{k_BT}\right)$

Topic-Thermodynamics and Statistical Mechanics

Sub topic: Partition function

Ans. (a)

Solution: From given Hamiltonian

$$\hat{H}=\begin{bmatrix} 0 & E_0 & 0 \\ E_0 & 0 & 2E_0 \\ 0 & 2E_0 & 0 \end{bmatrix}$$

Secular equation

$$\begin{vmatrix} 0-\lambda & E_0 & 0 \\ E_0 & 0-\lambda & 2E_0 \\ 0 & 2E_0 & 0-\lambda \end{vmatrix}=0$$

The eigen value of above Hamiltonian can be obtained as

$$\lambda=0,\sqrt{5}E_0,-\sqrt{5}E_0$$

For one particle system can be written as (micro canonical)

$$Z_1=e^{-0}+e^{-\beta\sqrt{5}E_0}+e^{-\beta\sqrt{5}E_0}$$

For two particle system can be written as (micro canonical)

$$Z_2=e^{-0}+e^{-\beta\sqrt{5}E_0}+e^{\beta\sqrt{5}E_0}$$

For two particle system can be written as (micro canonical)

$$Z_3=e^{-0}$$

Collection of all micro canonical is called grand canonical

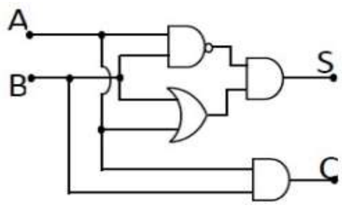
$$Z=Z_1+Z_2+Z_3=3+2e^{\beta\sqrt{5}E_0}+2e^{-\beta\sqrt{5}E_0}=3+4\cosh(\frac{\sqrt{5}E_0}{k_bT})$$

Q9. A half-adder circuit is defined as a two-input, two-output logic circuit where the output S gives the sum of inputs up to a single bit, and the output C gives carryover in a single bit.

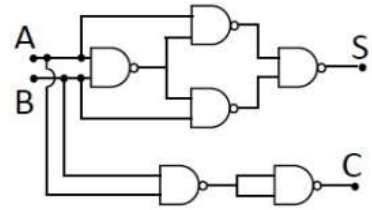
The expected truth table of the half-adder is given as

Which of the following circuits does NOT behave like a half adder?

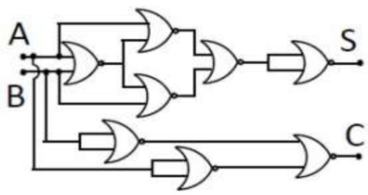
(a)



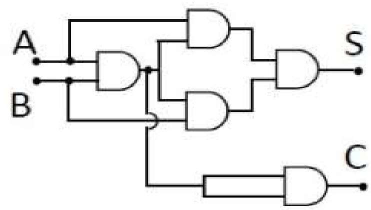
(b)



(c)



(d)



Topic-Electronics

Sub topic: Digital

Ans.: (a)

Solution: The sum of half adder is , $S = A\, XOR\, B$

The carry of half adder is, $C = A\ \ AND\, B$

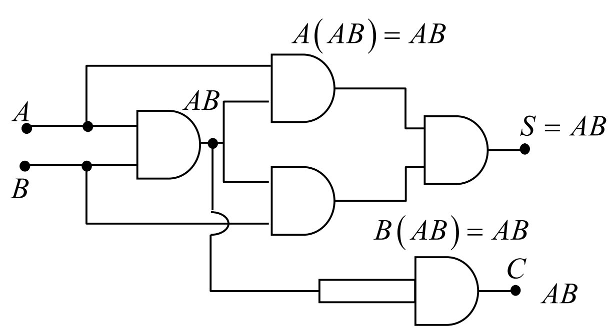
But from the configuration given in fig. (a) we can see

$S = A\ B$

$C = AB$

Which is not correctly providing same output as half adder

The following configuration is not correctly providing the output.



Q10. In an experiment that measures the angular distribution of the emission of particles, the angular distribution function is defined as

$$f(\theta) = \frac{n(\theta)}{n(\pi/2)}$$

Where $n(\theta)$ is the number of particles detected at an angle θ.

If, for a certain sample, we count

$$n(\pi/4) = 16265 n(\pi/2) = 8192$$

then the uncertainty

$$\left.\frac{\Delta f}{f}\right|_{\theta=\pi/4}$$

in this measurement at $\theta = \pi/4$ would be approximately

(a) 13.35% (b) 0.013% (c) 0.707% (d) 0.018%

Topic-Mathematical Physics

Sub topic: Differentiation

Ans.: (a)

Q11. The energy gap between the $n=1$ and the $n=2$ energy levels of a hydrogen atom is denoted E_0 .Now, consider a muonic carbon ion C^{5+} , i.e., a carbon nucleus (${}_6C^{12}$) orbited by a muon $\mu(q=-e, M_\mu = 210 m_e)$. The energy of the photon emitted in the transition of the muon from the $n=3$ level to the $n=2$ level of this ion will be approximately

(a) $1400E_0$ (b) $235E_0$ (c) $1050E_0$ (d) $7560E_0$

Topic-Atomic, Molecular and Laser Physics

Sub topic: Bohr's Theory

Ans.: (a)

Solution: $E_n{}' = \frac{\mu}{m_e} E_n \Rightarrow E_0 = 13.6 \times \frac{3}{4} \times \frac{1836}{1837} eV \Rightarrow 1eV = 4 \frac{\times 1837 \times E_0}{1836 \times 13.6 \times 3}$

$$E'_n(C^{12}) = \frac{\mu}{m_e} E_n \Rightarrow E = \frac{12 \times 1836 \times 208}{12 \times 1836 + 208} \times 13.6 \times \frac{36 \times 5}{36} eV$$

$$= \frac{12 \times 1836 \times 208}{12 \times 1836 + 208} \times 13.6 \times \frac{36 \times 5}{36} 4 \frac{\times 1837 \times E_0}{1836 \times 13.6 \times 3} = 1373 E_0$$

Q12. A lattice in the three-dimensional space has N sites, each occupied by an atom whose magnetic moment is μ. The lattice is in contact with a heat reservoir at a fixed temperature T. The atoms interact with an applied magnetic field

$$H = H(x)\hat{z}$$

Ignoring the interactions between the atoms, the average magnetic susceptibility per lattice site is given by

(a) $\frac{\mu^2}{3k_BT}$ (b) $\frac{\mu^2}{9k_BT}$ (c) $\frac{\mu}{3k_BT}$ (d) $\frac{\mu H}{3k_BT}$

Topic-Thermodynamics and Statistical Mechanics

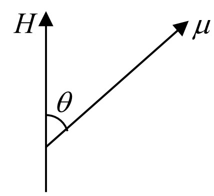
Sub topic: Partition Function

Ans. (a)

Solution: N Magnetic dipoles

μ Magnetic moment of each dipole

H External magnetic field



$$Z_1 = \int_{\infty}^{2\pi}\int_{0}^{\pi} e^{-\beta E}\sin\theta d\theta d\phi$$

$$Z_1 = 2\pi\int_0^{\pi} e^{+\beta\mu H\cos\theta}\sin\theta d\theta$$

Let

$$-\beta\mu H\cos\theta = t \Rightarrow \beta\mu H\sin\theta d\theta = dt$$

$$\therefore Z_1 = 2\pi\int_{-\beta\mu H}^{\beta\mu H}\frac{1}{-\beta\mu H}e^{-t}dt = \frac{+2\pi}{\beta\mu H}\left[e^{-t}\right]_{-\beta\mu H}^{\beta\mu H}$$

$$= \frac{2\pi}{\beta\mu H}\left[e^{\beta\mu H} - e^{-\beta\mu H}\right]$$

$$Z_1 = \frac{4\pi}{\beta\mu H}\sinh\beta\mu H$$

For N particles, partition function $= (Z_1)^N$

$$= \left(\frac{4\pi}{\beta\mu H}\sinh\beta\mu H\right)^N$$

Energy, $U = \frac{-1}{z}\frac{\partial z}{\partial \beta}$

$$U=\frac{-\beta\mu H}{4\pi\sinh\beta\mu H}\times\frac{4\pi}{\mu H}\left[\frac{\cos\beta\mu H\times\beta\mu H-\sinh\beta\mu H}{\beta}\right]$$

$$=\frac{1}{\beta}-\mu H\coth\beta\mu H$$

$$U=Nk_BT-N\mu H\coth\beta\mu H$$

$$F=-k_BT\ln Z=-Nk_BT\ln\frac{4\pi}{\beta\mu H}\sinh\beta\mu H$$

$$\langle M_Z\rangle=\left(-\frac{\partial F}{\partial H}\right)=-Nk_B\frac{T\partial}{\partial H}\left[\ln\frac{4\pi}{\beta\mu H}+\ln\sinh\beta\mu H\right]$$

$$=Nk_BT\left[\frac{\beta\mu H}{4\pi}\times\frac{-4\pi}{\beta\mu H}+\frac{\cosh\beta\mu H}{\sinh\beta\mu H}\cdot\beta\mu\right]$$

$$=Nk_BT\left[\frac{-1}{H}+\coth\beta\mu H\cdot\beta H\right]$$

$$\langle M_z\rangle=N\mu\left[\coth\beta\mu H-\frac{k_BT}{\mu H}\right]$$

Minimum value of $\langle M_z\rangle=\mu$

$$\langle M_z\rangle=\mu(x)=\mu^2\frac{H}{3k_BT}$$

$$\chi_H=\frac{\mu^2}{3k_BT}$$

Q13. The angular position of a star is found to change by an amount of 0.2 arc seconds (relative to the very distant background stars) when measured by a telescope on the Earth on two different nights separated by exactly six months. Note that the distance between the Earth and Sun is known to be approximately $1.5\times10^{13}\,cm$. If the energy flux received from the star is $F=10^{-7}\mathrm{ergs}^{-1}\,\mathrm{cm}^{-2}$ approximate value of its luminosity?

(a) $10^{33}\ ergs^{-1}$ (b) $10^{31}\ ergs^{-1}$ (c) $10^{35}\ ergs^{-1}$ (d) $10^{39}\ ergs^{-1}$

Topic-Classical Mechanics

Sub topic: STR

Q14. The binding energy ε_b of a nuclide $_Z X^A$ with atomic number and mass number A is given by the semi-empirical formula

$$\varepsilon_b = a_v A - a_s A^{2/3} - \frac{a_c Z(Z-1)}{A^{1/3}} + a_a \frac{(A-2Z)^2}{A}$$

where the constant parameters and source of effect for each term are

Volume term	Surface term	Coulomb term	Asymmetry term
a_v	a_s	a_c	a_a
15.56MeV	17.8MeV	0.7MeV	23.29MeV

For a spherical neutron star consisting of only neutrons and having uniform nuclear density throughout its volume, the Coulomb term is replaced by gravitational energy. What would be the smallest radius of this neutron star?

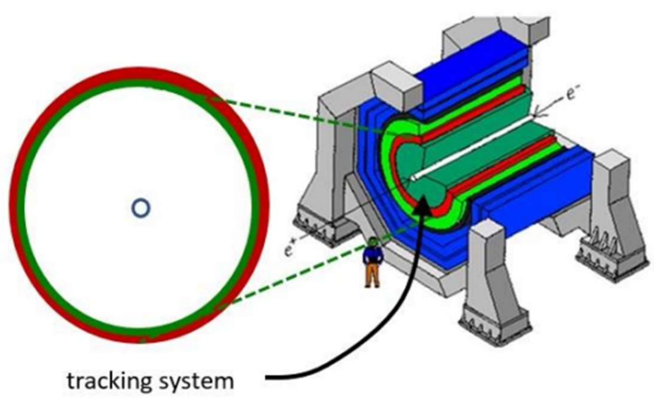
(a) $4.34\,km$ (b) $10.435\,km$ (c) $2.165\,km$ (d) $4.345\,km$

Topic-Nuclear and Particle Physics

Sub topic: Shell Model

Ans. (a)

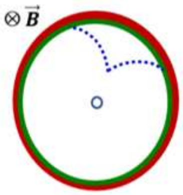
Q15. The figure below shows on the right a sketch of an electron-positron collider experiment where the innermost detector (shaded dark green) is a tracking system which records the tracks of charged particles which pass through it. On the left of the figure, a cross-sectional view of the same tracking system is shown. The narrow (white) pipe in the centre is where electrons and positrons are injected as shown and collide in the centre. (On the left it appears as a small central circle.) Inside the tracking system there is a strong uniform magnetic field collinear with the e^+ direction.

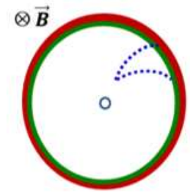


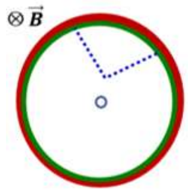
In one of the e^+e^- collisions, a high-energy $K_s^{\,0}$ meson is produced that subsequently decays as follows

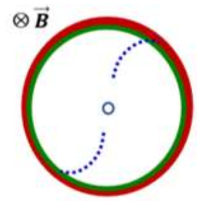
$$K_s^{\,0} \to \pi^+ + \pi^-$$

A possible representation of the tracks (dotted lines) of the pions $\pi^{\pm}$ in the tracking system would be

(a) 

(b) 

(c) 

(d) 

Topic-Nuclear and Particle Physics

Sub topic: Particle Physics

Ans.: (a)

Solution: The radius in cyclotron motion is $R = \dfrac{mv}{qB}$

Since, π^+ and π^- are oppositely charged they will rotate in different direction. If π^+ rotates clockwise then π^- rotates anticlockwise or vice versa