TIFR – 2022 [Solution]

Q1. Consider a square which can undergo rotations and reflections about its centre, where making no transformation at all is counted as a rotation by 0^0. The total number of such distinct rotations and reflections which will keep the square unchanged is

(a) 8　　　　　　　(b) 4　　　　　　　(c) 16　　　　　　　(d) 32

Topic-Solid state Physics

Sub topic-Crystal symmetry

Ans.: (a)

Q2. Consider the two-dimensional polar integral

$$P = \int dr d\theta r^{19} e^{-r^2} \sin^8 \theta \cos^{11} \theta$$

If the integration is over only the first quadrant $(0 \le \theta \le \pi/2)$, the value of P is

(a) 180　　　　　　(b) 88π　　　　　　(c) 20160　　　　　　(d) 16π

Topic-Mathematical Physics

Sub topic-Gamma function

Ans. (a)

Solution: The two-dimensional integration can be written as follows

$$P = \int dr d\theta r^{19} e^{-r^2} \sin^8 (\theta) \cos^{11}(\theta)$$

Consider $r^2 = t \Rightarrow 2r dr = dt \Rightarrow r dr = \dfrac{dt}{2}$

$$P = \int dr d\theta r^{19} e^{-r^2} \sin^8(\theta) \cos^{11}(\theta) = \frac{\int t^9 e^{-t} dt}{2} \int \sin^8(\theta) \cos^{11}(\theta) d\theta = \frac{9!}{4} \frac{\overline{|9/2}\,\overline{|12/2}}{\overline{|21/2}}$$

$$= \frac{9!5!}{4} \frac{\overline{|9/2}}{\overline{|21/2}} = \frac{9!5!}{4} \frac{\overline{|9/2}}{\dfrac{19}{2} \times \dfrac{17}{2} \times \dfrac{15}{2} \times \dfrac{13}{2} \times \dfrac{11}{2} \times \dfrac{9}{2} \times \overline{|9/2}} = 111$$

Q3. Consider a set of three 3-dimensional vectors

$$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

These vectors undergo a linear transformation

$$A \to A' = MA \qquad B \to B' = MB \qquad C \to C' = MC$$

Where M is given by

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$$M = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

The volume of a parallelopiped whose sides are given by the transformed vectors A', B' and C' is

(a) 8   (b) 4   (c) 2   (d) 16

Topic-Mathematical Physics

Sub topic-Matrices

Ans. (a)

Solution: A set of three 3-dimensional vectors is given,

$$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

These vectors undergo a linear transformation,

$$A \to A' = \mathbb{M}A, \ B \to B' = \mathbb{M}B, \ C \to C' = \mathbb{M}C$$

where $\mathbb{M}$ is given by

$$\mathbb{M} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$B' = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$C' = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 5 \end{pmatrix}$$

The volume of a parallelopiped whose sides are given by the transformed vectors A', B' and C' is,

$$\det \begin{pmatrix} 1 & 1 & 10 \\ 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix} = 8$$

Q4. A solid homogeneous sphere floats in water with a portion sticking out above the water, as shown in the figure below. The height of the highest point above the water surface is $R/2$ where R is the radius of the sphere.
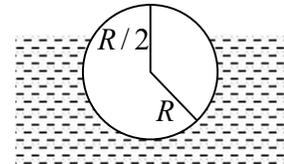
If the density of water is $1 g\, cm^{-3}$, the density of the material (in $g\, cm^{-3}$) must be

(a) $27/32$ (b) $5/32$ (c) $13/18$ (d) $5/18$

Topic-Mechanics

Sub topic-Fluid mechanics

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Ans. : (a)

Solution: According to Archimedes principle, the weight of a sphere is equal to the weight of water displaced by submerged water

$\rho_s V_s g = \rho_w V_w g$ where volume of sphere $V_S = \dfrac{4\pi}{3}R^3$ volume of water displaced by submerged

sphere $V_w = \dfrac{4\pi}{3}R^3 - \dfrac{2\pi}{8}R^3 = \dfrac{32\pi R^3 - 6\pi R^3}{24} = \dfrac{26\pi R^3}{24}$

$\rho_s \dfrac{4\pi R^3}{3}g = \rho_w \dfrac{26\pi R^3}{24}g \Rightarrow \rho_s = \dfrac{26}{32}\rho_w$ hence water is above $R/2$ so best answer is $27/32$

Q5. A particle of mass m moves under the action of a central potential

$$V(r) = -\frac{e^2}{r}$$

where e is a constant. Two vectors which remain conserved during the motion are

(i) the angular momentum $\vec{L} = \vec{r} \times \vec{p}$

(ii) the Runge-Lenz vector $\vec{K} = \vec{p} \times \vec{L} - me^2 \hat{r}$ (where $\hat{r} = \vec{r}/r$)

The conserved energy E of the particle can be written as

(a) $\dfrac{K^2 - m^2 e^4}{2mL^2}$ (b) $\dfrac{m^2 e^4 - K^2}{2mL^2}$ (c) $\dfrac{2mL^2}{K^2 - m^2 e^4}$ (d) $\dfrac{2mL^2}{m^2 e^4 - K^2}$

<div align="right">

Topic-Mechanics

Sub topic-Cetral force problem
</div>

Ans.: (a)

Solution: angular momentum vector $\vec{L} = \vec{r} \times \vec{p}$ and $\vec{K} = \vec{p} \times \vec{L} - me^2 \hat{r}$

$\vec{K}.\vec{L} = 0$ where $\vec{L}$ is perpendicular to $\vec{p} \times \vec{L}$ and $\vec{r}$

$\vec{K}.\vec{r} = (\vec{p} \times \vec{L}).\vec{r} - me^2 r = \vec{K}.\vec{r} = L.(\vec{r} \times \vec{p}) - me^2 r = l^2 - me^2 r$

$kr\cos\theta = l^2 - me^2 r \Rightarrow \dfrac{1}{r} = \dfrac{me^2}{l^2}\left(1 + \dfrac{k}{me^2}\cos\theta\right)$

It is equivalent to equation of orbit so $\dfrac{k}{me^2}$ is equivalent to eccentricity which is given by

$\dfrac{k}{me^2} = \sqrt{1 + \dfrac{2El^2}{m.(e)^2}} = \dfrac{k^2}{m^2 e^4} = \dfrac{me^2 + 2El^2}{me^4} \Rightarrow mk^2 = me^2 + 2El^2$

So $E = \dfrac{k^2 - me^4}{2ml^2}$

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Q6. The Principle of Linear Superposition of electron states in quantum mechanics is nicely illustrated by the

(a) Davisson-Germer experiment

(b) Compton scattering experiment

(c) Franck-Hertz experiment

(d) Millikan oil-drop experiment

Topic-Quantum mechanics

Sub topic- Davisson Germer experiment

Ans. (a)

Solution: Davisson Germer effect will explain the wave nature of electrons which can be explained from the superposition of electrons.

Q7. A particle moves in one-dimension x under the influence of a potential $V(x)$ as sketched in the figure below. The shaded region corresponds to infinite V, i.e., the particle is not allowed to penetrate there.
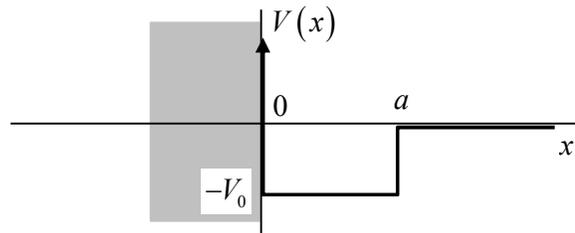
If there is an energy eigenvalue $E = 0$, then a and V_0 are related by

(a) $a^2 V_0 = \dfrac{\left(n + \dfrac{1}{2}\right)^2 \pi^2}{2m}$

(b) $a^2 V_0 = \dfrac{n^2 \pi^2}{2m}$

(c) $a^2 V_0 = \dfrac{\left(n + \dfrac{1}{2}\right) \pi^2}{2m}$

(d) $a^2 V_0 = \dfrac{n \pi^2}{2m}$

Topic-Quantum mechanics

Sub topic- Finite potential well

Ans.: (a)

Solution: The bound state can be given by $\eta^2 + \xi^2 = R^2$ and $\eta = -\xi \cot \xi$ where

$$\xi = \sqrt{\frac{2m(E + V_0)}{\hbar^2}}\, a$$

and $\eta = \gamma a = \sqrt{-\dfrac{2mE}{\hbar^2}}\, a$

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if energy $E=0$ for energy eigenvalue $\xi = \left(n+\dfrac{1}{2}\right)\pi \Rightarrow \sqrt{\dfrac{2m(0+V_0)}{\hbar^2}}a = \left(n+\dfrac{1}{2}\right)\pi$ for $E=0$ so

$$V_0 a^2 = \frac{\left(n+\dfrac{1}{2}\right)^2 \pi^2}{2m}$$

Q8. In a hydrogenic atom of atomic number Z, the probability amplitude that the nucleus will capture an electron from its own K-shell is proportional to the overlap between the nuclear wave-function

$$\psi_n(\vec{r}) = \frac{1}{\sqrt{8\pi r_N^3}} e^{r/r_N}$$

And the electron wave-function

$$\psi_e(\vec{r}) = \frac{Z^{3/2}}{\sqrt{8\pi a_0^3}} e^{-Zr/a_0}$$

Where a_0 is the Bohr radius and r_N is the nuclear radius, which is known to vary as $r_N \propto Z^{0.37}$.

The probability of electron capture, to a vary good approximation, will be proportional to Z^α where α is

(a) 4.11   (b) 2.22   (c) 2.05   (d) 1.11

Topic-Nuclear Physics

Sub topic- Radioactivity

Ans.: (a)

Solution: $\psi_n = \dfrac{1}{\sqrt{8\pi r_N^3}} e^{r/r_N}$

$$\psi_e = \frac{Z^{3/2}}{\sqrt{8\pi a_0^3}} e^{-Zr/a_0}$$

The required probability can be written as follows

$$P = \left|\langle \psi_N || \psi_e \rangle\right|^2 = \left|\frac{Z^{3/2}}{\sqrt{8\pi a_0^3}} \int_0^\alpha e^{-Zr/a_0} \frac{1}{\sqrt{8\pi r_N^3}} e^{-r/r_N} r^2 dr \int_{\theta=0}^\pi \sin(\theta)d\theta \int_{\phi=0}^{2\pi} d\phi\right|^2 = Z^{4.11}$$

Given that $r_N = kZ^{0.37}$

Thus, $P = \left|\dfrac{Z^{3/2}}{8\pi a_0^{3/2}} 4\pi(kz^{0.37})^{-3/2} \int_0^\alpha r^2 e^{-(Z/a_0 - \frac{1}{r_N})^2}\right|^2$

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$$P\alpha \left| Z^{0.37 \times -\frac{3}{2}+\frac{3}{2}} \frac{2!}{(\frac{Z}{a_0} - \frac{1}{kz^{0.37}})^3} \right|^2 \alpha \left| Z^{0.94+3 \times 0.37} \right|^2 \alpha Z^{4.11} = Z^\alpha \rightarrow \alpha = 4.11$$

Q9. Treat the hydrogen molecule H_2 as a rigid rotator. The next-to-largest wavelength in its rotational spectrum is about $111 \mu m$. From this it can be estimated that the separation between the pair of hydrogen atoms is about

(a) $0.12 nm$ (b) $24.4 nm$ (c) $64.4 nm$ (d) $3.07 \mu m$

Topic-Atomic, Molecular and Laser

Sub topic- Molecular Physics

Ans.: (a)

Solution: The energy of rigid rotator can be written as follows

$$E_J = \frac{\hbar^2}{2I} J(J+1); \quad J = \text{Rotational Quantum number} = 0,1,2,3,.....$$

$I = $ Moment of inertia $= \mu r_0^2$

The selection rule for rotational transition is $\Rightarrow \Delta J = \pm 1$.

Thus, the energy of emitted photon during transition from J to $J-1$ can be written as follows

The minimum energy-maximum wavelength of emitted photon will be for $J=0$

The second largest wavelength will be for $J=1$

$$\Delta E_j = E_2 - E_1 = \frac{2\hbar^2}{I} = \frac{hc}{\lambda} \Rightarrow \frac{2\hbar^2}{I} = \frac{hc}{\lambda} \Rightarrow \frac{2\hbar^2}{\mu r_0^2} = \frac{hc}{\lambda} \Rightarrow \mathbf{r_0} = \sqrt{\frac{2\lambda\hbar^2}{c\mu h}}$$

$$\Rightarrow \mu = \frac{m_H \times m_H}{2m_H} = \frac{m_H}{2} \Rightarrow r_0 = \sqrt{2\frac{111 \times 10^{-6} \times (1.05 \times 10^{-34})^2}{\frac{938 \times 10^6 \times 1.6 \times 10^{-9}}{2} \times 6.63 \times 10^{-34} \times 3 \times 10^8}} = 0.12 \, nm$$

Q10. A falling raindrop, spherical in shape, with a diameter of $1 \mu m$, acquires a uniform negative charge due to friction with air. The electric field at a distance of $10 \mu m$ from the surface of the droplet is measured to be $101 Vm^{-1}$.

(a) 7 (b) 7.02×10^6 (c) 1.4×10^{23} (d) 1414

Topic: EMT

Sub topic- Electrostatic

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Ans. (a)

Solution: Radius of drop

$$\Rightarrow R = \frac{D}{2} = 0.5\,\mu m$$

The electric field at a distance of $r = 10.5\,\mu m$ from the centre of drop is

$$E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{9\times10^9 \times ne}{\left(10.5\times10^{-6}\right)^2} = 101$$

$$\frac{9\times10^9 \times n \times 1.6\times10^{-19}}{\left(10.5\times10^{-6}\right)^2} = 101 \Rightarrow 13n = 101 \Rightarrow n = 101/13 = 7$$

Q11. An electromagnet is made by winding N turns of wire around a wooden cylinder of diameter d and passing a current I through it. When the current flows, a magnetic field of magnitude B is produced at a perpendicular distance z_0 from the axis of the cylinder, where $z_0 \gg d$.

If the number of turns N, the diameter of the wooden cylinder d and the current I are all doubled, then the magnitude of the magnetic field will be $B/2$ at a distance $z =$

(a) $3.2z_0$ (b) $0.5z_0$ (c) $4.8z_0$ (d) $2.4z_0$

Topic-EMT

Sub topic- Magnetostatic

Ans.: (a)

Solution: The magnetic field can be written as follows

$$B = \frac{\mu_0}{4\pi}\frac{m}{r^3}, \quad B = \frac{\mu_0}{4\pi}\frac{NIA}{Z_0^{\,3}}, \quad B = \frac{\mu_0}{4\pi}\frac{NI\pi d^2}{4Z_0^{\,3}}$$

$$N \to 2N, d \to 2d, \; I \to 2I$$

$$B' = \frac{\mu_0}{4\pi}\frac{16NI\pi d^2}{4Z^3} = B/2 \Rightarrow \frac{\mu_0}{4\pi}\frac{16NId^2}{4Z^3} = \frac{\mu_0}{8\pi}\frac{NI}{4Z_0^{\,3}}\pi d^2 \to Z = 3.2Z_0$$

Q12. If an electron is set into oscillatory motion by the electric field of a laser of intensity $150\,Wm^{-2}$ and wavelength $554\,nm$, the amplitudes of its displacement and velocity, respectively, are expected to be

(a) $5.1\times10^{-18}\,m, 1.7\times10^{-2}\,ms^{-1}$ (b) $3.4\times10^{-17}\,m, 1.0\times10^{-1}\,ms^{-1}$

(c) $3.4\times10^{-16}\,m, 1.7\times10^{-1}\,ms^{-1}$ (d) $3.4\times10^{-18}\,m, 1.7\times10^{-2}\,ms^{-1}$

Topic-Oscillation, waves and optics

Sub topic- SHM

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Ans.: (a)

Solution: From the equation of motion, we can write

$$ma = m\frac{dv}{dt} = qE = qE_0\sin(\omega t - kz)$$

$$\frac{dv}{dt} = \frac{qE_0}{m}\sin(\omega t - kz) \Rightarrow \int dv = \int \frac{qE_0}{m}\sin(\omega t - kz)dt$$

$$\Rightarrow v = -\frac{qE_0}{m\omega}\cos(\omega t - kz) = \frac{eE_0}{m\omega}\cos(\omega t - kz)$$

The maximum speed can be written as follows

$$v = \frac{1.6\times10^{-19}}{m\omega}E_0$$

The intensity of laser light can be written as follows

$$I = \frac{1}{2}c\varepsilon_0 E_0{}^2 \Rightarrow E_0 = \sqrt{\frac{2I}{c\varepsilon_0}} = \sqrt{\frac{2\times150}{3\times10^8\times8.85\times10^{12}}} = 3.36\times10^2\ N/C$$

$$\omega = 2\pi v = 2\pi\frac{c}{\lambda} = 2\times3.14\frac{3\times10^8}{554\times10^{-9}} = 3.4\times10^{15}/s$$

The velocity amplitude can be written as follows

$$v = \frac{eE_0}{m\omega} = \frac{1.6\times10^{-19}\times3.36\times10^2}{9.1\times10^{-31}\times3.4\times10^{15}} = 0.17\times10^{-17+16}\ m/s = 1.7\times10^{-2}\ m/s$$

$$\frac{dx}{dt} = \frac{eE_0}{m\omega}\cos(\omega t - kz) \Rightarrow \int dx = \int\frac{eE_0}{m\omega}\cos(\omega t - kz)dt$$

$$x = \frac{eE_0}{m\omega^2}\sin(\omega t - kz) = A\sin(\omega t - kz),\ A = \frac{eE_0}{m^2\omega} = -\frac{1.6\times10^{-19}\times3.36\times10^2}{9.1\times10^{-31}\times\left(3.4\times10^{15}\right)^2} = 5.11\times10^{-18}\ m$$

Q13. A bicycle tyre is pumped with air to an internal pressure of 6 atm at $20^0 C$, at which point it suddenly bursts. Assuming the external pressure to be 1 atmosphere and the subsequent sudden expansion to be adiabatic, the temperature immediately after the burst is approximately

(a) $-97.5^0 C$ (b) $-108.5^0 C$ (c) $45.5^0 C$ (d) $216.0^0 C$

Topic-Thermodynamics and statistical
Sub topic- Thermodynamics

Ans.: (a)

Solution: The process is adiabatic

So, we can write

$$P_1^{1-\gamma}T_1^{\gamma} = P_2^{1-\gamma}T_2^{\gamma},\ P_1^{1-\gamma}T_1^{\gamma} = P_2^{1-\gamma}T_2^{\gamma} \Rightarrow T_2 = \left(\frac{P_1}{P_2}\right)^{\frac{1-\gamma}{\gamma}}T_1 = 6^{\frac{1-1.4}{1.4}}\times293 = 175\,K = -97.59\,{}^O C$$

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Q14. A vertical cylinder of height H is filled with an ideal gas of classical point particles each of mass m and is allowed to come to equilibrium under gravity at a temperature T. The mean height of these particles is

(a) $\dfrac{k_B T}{mg}\left(1 - \dfrac{mgH/k_B T}{e^{mgH/k_B T} - 1}\right)$

(b) $\dfrac{H}{3}\dfrac{mgH/k_B T}{e^{mgH/k_B T} + 1}$

(c) $\dfrac{k_B T}{mg}\left(1 - \dfrac{2mgH/k_B T}{e^{mgH/k_B T} + 1}\right)$

(d) $\dfrac{H}{3}\dfrac{mgH/k_B T}{e^{mgH/k_B T} - 1}$

Topic: Thermodynamics and statistical

Sub topic: statistical

Ans.: (a)

Solution: The energy can be written as follows

$$\varepsilon = mv^2 + mgz$$

The mean hit can be write

$$\langle z \rangle = \iint dx\, dy \int_0^H dz \int_{-\infty}^{\infty} d^3 v\, e^{-\beta\left(\frac{mv^2}{2} + mgz\right)}$$

$$Z = A\int_0^H e^{-\beta mgz)} dz \int_{-\infty}^{\infty} d^3 v\, e^{-\beta\left(\frac{mv^2}{2}\right)} = \frac{A}{mg\beta}\left(\frac{2\pi}{m\beta}\right)^{3/2}\left[1 - e^{-mgH\beta}\right]$$

The probability distribution can be written as follows

$$\rho = \frac{e^{-\beta E}}{Z}$$

$$\langle z \rangle = \frac{A\int_0^H z e^{-\beta mgz)} dz \int_{-\infty}^{\infty} d^3 v\, e^{-\beta\left(\frac{mv^2}{2}\right)}}{Z} = \frac{A}{Z}\left(\frac{2\pi}{m\beta}\right)^{3/2}\left[\frac{Z e^{-mgH\beta}}{-(mg\beta)} - \int \frac{e^{-mgH\beta}}{-(mg\beta)} dz\right]$$

$$= \frac{A}{Z}\left(\frac{2\pi}{m\beta}\right)^{3/2}\left[\frac{Z e^{-mgH\beta}}{-(mg\beta)} - \frac{e^{-mgH\beta}}{(mg\beta)^2}\right] = \frac{A}{Z}\left(\frac{2\pi}{m\beta}\right)^{3/2}\left[H\frac{e^{-mgH\beta}}{-mg\beta} - \frac{e^{-mgH\beta}}{(mg\beta)^2} + \frac{1}{(mg\beta)^2}\right]$$

$$= \frac{A\left(\frac{2\pi}{m\beta}\right)^{3/2}\frac{1}{\beta m\rho}\left[\frac{1 - e^{-mgH\beta}}{\beta mg} - H e^{-mgH\beta}\right]}{A\left(\frac{2\pi}{m\beta}\right)^{3/2}\frac{1}{\beta mg}(1 - e^{-\beta mgH})}$$

$$= \frac{\left(1 - e^{-mgn\beta}\right)}{\beta mg\left(1 - e^{-mg + n\beta}\right)} - \frac{H e^{-mgH\beta}}{1 - e^{-\beta mgH}} = \frac{kT}{mg}\left[1 - \frac{mgH/kT}{e^{mgH/kT} - 1}\right]$$

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$$= \frac{\frac{A}{(m\beta g)}\left(\frac{2\pi}{m\beta}\right)^{3/2}\left[\frac{1-e^{-mgH\beta}}{(mg\beta)}+He^{-mgH\beta}\right]}{A\left(\frac{2\pi}{m\beta}\right)^{3/2}\frac{1}{m\beta g}\left(1-e^{-\beta mgH}\right)} = \frac{1}{m\beta g}\left[1-\frac{mg\beta He^{-mgH\beta}}{\left(1-e^{-\beta mgH}\right)}\right]$$

$$\langle z\rangle = \frac{k_B T}{mg}\left[1-\frac{mgH\beta}{\left(e^{-mgH/k_B T}-1\right)}\right]$$

Q15. Two students A and B, measure the time period of a simple pendulum in the laboratory using the same stopwatch but following two different methods.

— Student A measures the time taken for one oscillation and repeats it for N_A number of times and finds the average.

— Student B, on the other hand, measures the time taken for N_B number of oscillations and then computes the period.

Given that $N_A, N_B \gg 1$, to ensure that both students measure the time period with the same uncertainty, the relation between N_A and N_B must be

(a) $N_A = N_B^2$ (b) $N_A = \sqrt{N_B}$ (c) $N_A = N_B$ (d) $\ln 2N_A = N_B$

Topic-Mechanics

Sub topic- Oscillation

Ans.: (a)

Q16. A commercial advertisement for a solar power converter claims that when the temperature of the plate (area $1.6m^2$) absorbing 20% of the solar energy (solar constant is about $1.36kWm^{-2}s^{-1}$) reaches $127^0 C$ and the rest of the device is at room temperature ($27^0 C$), the system will deliver a power of $100W$.

If a prospective customer comes to you for advice about buying this device, your advice should be that

(a) it is an efficient device for the given specifications.

(b) the power delivered is very small for the given specifications.

(c) the advertisement is false and the device cannot deliver so much power.

(d) other similar devices are available which can deliver $1.5-2.0$ times the power with the same specifications.

Topic-Solid state Physics

Sub topic: Semiconductor Physics

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Ans. (a)

Solution: It is an efficient device for the given specifications.

Q17. Since the refractive index of water is $4/3$, the angular velocity (in degrees per hour) of the Sun at noon is perceived by a fish in the ocean deep below the surface as around

(a) 11.3 (b) 15.0 (c) 13.2 (d) 20.0

Topic-Oscillation, waves and optics

Sub topic- Optics

Ans. : (a)

Solution: $\dfrac{\sin(i)}{\sin(r)} = \dfrac{\mu_2}{\mu_1} = \dfrac{\frac{4}{3}}{1} \Rightarrow \sin(i) = \dfrac{4}{3}\sin(r)$
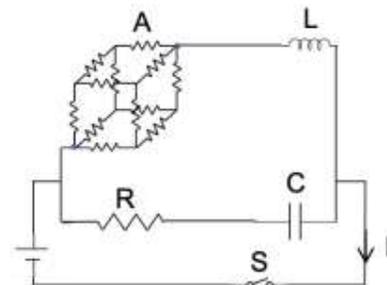
$\cos(i)\dfrac{di}{dt} = \dfrac{4}{3}\cos(r)\dfrac{dr}{dt}$, $\cos(i)\omega_g = \dfrac{4}{3}\cos(r)\omega_f$

An normal incidence

$i = r = 0^0$

$\omega_f = \dfrac{3}{4}\omega_g$, $\omega g = \dfrac{360}{204} = 15^o/Hr$, $\omega_f = 15 \times \dfrac{3}{44} = 11.3^o/Hr$

Q18. The circuit diagram on the right shows a block A representing a cubic structure comprising 12 identical resistances of 120Ω each, whose body diagonal vertices are connected to the rest of the circuit with an inductor $L = 10mH$, a resistor $R = 100\Omega$, and a capacitor $C = 1\mu F$.

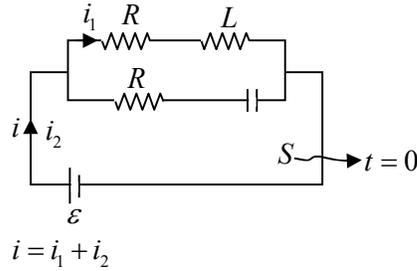Now, the switch S is turned on at $t = 0$. The earliest time at which the current reaches a steady value I_0 is

(a) Zero (b) $100\mu s$ (c) $200\mu s$ (d) Infinite

Topic-Electronics

Sub topic- Circuit analysis

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Ans. : (a)

Solution:

$$i = i_1 + i_2$$

The equivalent resistant can be written as follows

$$= \frac{5R}{6} = \frac{5 \times 120}{6} = 100\Omega$$

$$i = i_1 + i_2$$

$$\varepsilon - i_1 R - L\frac{d_1}{dt} = 0$$

$$\frac{di_1}{dt} + \frac{R}{L}i_1 = \frac{\varepsilon}{L}$$

$$i_1 = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right)$$

$$\varepsilon - \frac{q}{c} - i_2 R = 0, \; i_2 = \frac{dq}{dt}$$

$$\varepsilon - \frac{q}{c} - \frac{dq}{dt}R = 0, \frac{dq}{dt} + \frac{q}{RC} = \frac{\varepsilon}{R}$$

$$q = c\varepsilon\left(1 - e^{-t/RC}\right)$$

$$i_2 = \frac{dq}{dt} = \frac{\varepsilon}{R}e^{-t/Rc}$$

$$i = i_1 + i_2 = \frac{\varepsilon}{R}e^{-t/Rc} + \frac{\varepsilon}{R}\left(1 - e^{-tR/c}\right) = \frac{\varepsilon}{R}\left[e^{-t/Rc} + 1 - e^{-tR/c}\right]$$
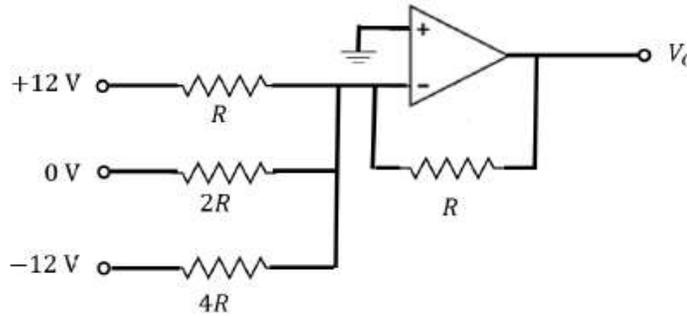
$$C = 1\mu f = 10^{-6}f$$
$$L = 10mH = 10 \times 10^{-3} = 10^{-2}H$$

$$i = \frac{\varepsilon}{100}\left[e^{-t\times10^{+4}} - e^{-t\times10^4} + 1\right], i = \frac{\varepsilon}{10}$$

Which implies it is already in steady state at $t = 0$.

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Q19. Consider a circuit with an operational amplifier (op amp) and four resistors as sketched below.

The output voltage V_0 is

(a) $-9V$ 　　　　(b) $0V$ 　　　　(c) $-12V$ 　　　　(d) $-6V$

Ans.: (a)

Solution: The output voltage for inverting amplifier can be written as follows

$$V_o = -\left(\frac{R_f}{R_1} V_{in} + \frac{R_f}{R_2} V_{in} + \frac{R_f}{R_3} V_{in} \right)$$

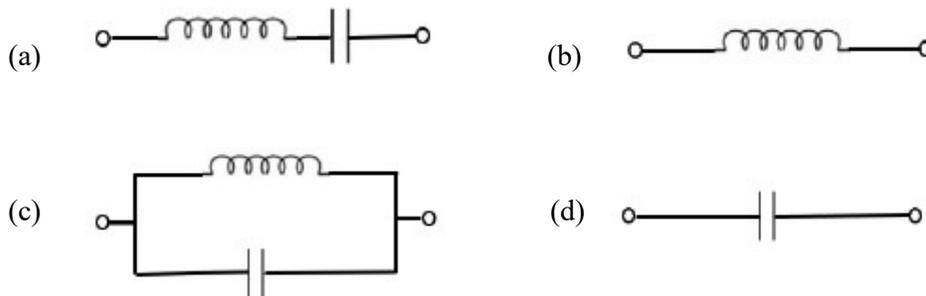$$V_o = -\left(\frac{R}{R} 12 - \frac{R}{4R} 12 \right) = -\left(\frac{R}{R} 12 - \frac{R}{4R} 12 \right) V = -9V$$

Q20. It is required to design a circuit with an impedance $Z(\omega)$ such that

$$Z(\omega) = ik(\omega - \omega_0)$$

for a range of frequencies ω such that $|\omega - \omega_0| / \omega_0 \ll 1$

where k and ω_0 are constant real numbers.

A possible design for this circuit would correspond to

(a) 　　　　(b)

(c) 　　　　(d)

Ans.: (a)

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Solution: Let us consider the impedance of the series LC circuit. The total impedance is given by the sum of the inductive and capacitive impedances:

$Z = Z_L + Z_C$

Writing the inductive impedance as $Z_L = j\omega L$ and capacitive impedance as $Z_C = 1/j\omega C$ and substituting gives

$$Z(\omega) = j\omega L + 1/j\omega C$$

$$Z(\omega) = j(\frac{\omega^2 - 1}{\omega C})$$

Resonance frequency $\omega_0 = \frac{1}{\sqrt{LC}}$

$Z(\omega) = jL\ (\frac{\omega^2 - \omega_0^2}{\omega}), Z(\omega) = jL\ \frac{(\omega + \omega_0)(\omega + \omega_0)}{\omega}, Z(\omega) = jL\ \frac{(\omega - \omega_0)(\omega + \omega_0)}{\omega}$

For frequency range

$Z(\omega) = 2jL\ (\omega - \omega_0) = jk(\omega - \omega_0)$

Q21. On a wet monsoon day at 12 noon, a thin film of oil of thickness $0.3\mu m$ is formed on a wet road. If the refractive index of oil and water are 1.475 and 1.333, respectively, which of the following wavelengths of light will be reflected with maximum intensity?

(a) 590nm   (b) 407nm   (c) 443nm   (d) 640nm

Topic-Oscillation, waves and optics

Sub topic- Optics

Ans.: (a)

Solution: $2t + \frac{\lambda}{2} = m\lambda$

$2t = m\lambda - \frac{\lambda}{2}, \ \lambda = \frac{2t}{m - \frac{1}{2}}; \ m = 2 \Rightarrow \lambda = \frac{2 \times 0.3 \times 1.47}{2 - \frac{1}{2}} = 590\,nm$

Q22. A gas of atoms, each of mass m, in thermal equilibrium at a temperature T, is radiating with a frequency v_0. The Doppler broadening (full width at half maximum, or FWHM) of the observed spectral line would be given by

(a) $\frac{2v_0}{c}\sqrt{\frac{2\ln 2k_B T}{m}}$ (b) $\frac{v_0}{c}\sqrt{\frac{2k_B T}{m}}$ (c) $\frac{2v_0}{c}\sqrt{\frac{\ln 2k_B T}{m}}$ (d) $\frac{2v_0}{c}\sqrt{\frac{2k_B T}{m}}$

Topic-Atomic, Molecular and Laser Physics

Sub topic- Atomic Physics

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Ans. : (a)

Solution: The doppler broadening arises due to random motion of gasses molecules which are source of photon. Due to the random motion the broadening in the spectral line can be written as follows

$$v - v_0 = \frac{v_0}{c} V$$

Where V is the speed of gas molecules. The V can be considered as the most probable speed which will be

$$V = \sqrt{\frac{2k_B T}{m}}$$

The intensity distribution can be written as follows

$$I(\omega) = I_0 \exp\left[\frac{-m_0 c^2 (\omega_0 + \omega)^2}{2kT\omega_0^2}\right]$$

Thus, the broadening can be defined as

$$I(\omega) = \frac{I_0}{2}, \quad \omega_0 - \omega = \Delta\omega$$

$$I_0/2 = I_0 \exp\left[\frac{-m_0 c^2 (\omega_0 - \omega)^2}{kT\omega_0^2}\right]$$

Which can further be written as

$$\Delta\omega = 2.35 \frac{\omega_0}{c}\sqrt{\frac{2k_B T}{m}} = \frac{v_0}{c}\sqrt{\frac{2\ln 2 k_B T}{m}}$$

Q23. Two particles, as specified in the table below, both enter a region of uniform magnetic field in a direction perpendicular to the field direction.

Particle	Rest Mass	Kinetic Energy
Alpha	3.7GeV	11.2GeV
Deutron	1.9GeV	20.0MeV

If both the particles then follow circular trajectories in the magnetic field, the ratio of their time periods for one full revolution must be

(a) 4.0   (b) 3.0   (c) 2.0   (d) 1.0

Topic-EMT

Sub topic- Magnetostatic

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Ans.: (a)

Solution: The time period of circular motion can be written as follows

$$T = \frac{2\pi m_0 \gamma}{qB} \Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}; \; T = mc^2 - m_0 c^2 \Rightarrow \gamma = 1 + \frac{T}{m_0 c^2}$$

$$T_\alpha = \frac{2\pi m_\alpha c^2}{2eB}\left(1 + \frac{T}{m_\alpha c^2}\right) = \frac{2\pi}{2eB}\left(m_\alpha c^2 + T\right), \; T_\alpha = \frac{2\pi}{2eB}\left(m_\alpha c^2 + T\right), \; T_D = \frac{2\pi}{eB} m_D c^2$$

$$\frac{T_\alpha}{T_D} = \frac{2\pi}{2eB}\left(m_\alpha c^2 + T\right)\frac{1}{\frac{2\pi}{eB} m_D c^2} = \frac{\left(m_\alpha c^2 + T\right)}{m_D c^2} = \frac{3.7 + 11.2}{1.9} = 4$$

Q24. Natural potassium contains a radioactive component of ^{40}K that has two decay modes.

— In the first mode, ^{40}K undergoes a β decay to the ground state of ^{40}Ca.

— In the second mode, ^{40}K undergoes an electron capture to the excited state of

^{40}Ar, followed by a single γ transition to the ground state of ^{40}Ar.

The amount of radioactive ^{40}K in a natural potassium (atomic weight of 39.089) sample is known

to be 0.0118 percent. It is also known that in the decay of ^{40}K, for every

100β particles emitted, there number of γ-photons emitted is 12.

If the number of β-particles emitted per second by $1kg$ of natural potassium is 2.7×10^4, the mean

lifetime of ^{40}K in years is

(a) 1.9×10^9 (b) 1.3×10^9 (c) 1.7×10^9 (d) 1.1×10^8

Topic: Nuclear and particle Physics

Subtopic: Radioactivity

Ans.: (a)

Q25. The free electron model of metals (Drude model) explains several physical properties, but cannot be used to explain

(a) positive value of Hall coefficient   (b) magnetic susceptibility of the metal

(c) electrical conductivity of the metal   (d) thermal conductivity of the metal

Topic- Solid state Physics

Sub topic-Hall effect

Ans.: (a)

Solution: positive values of Hall coefficient for metals can be explained using drude theory.

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Section B

(Only for Integrated M.Sc.-Ph.D. candidates)

This Section consists of 15 questions. All are of multiple-choice type. Mark only one option on the online interface provided to you. If more than one option is marked, it will be assumed that the question has not been attempted. A correct answer will get +5 marks, an incorrect answer will get 0 mark.

Q26. The value of the integral

$$\int_{-\pi/2}^{+\pi/2} dx \cosh kx^2 \sin^2 x$$

In the large - k limit, will be

(a) $\dfrac{1}{k\pi} e^{k\pi^2/4}$ 　　　(b) $\cos\left(\dfrac{\pi^2}{4}\right)$ 　　　(c) $\dfrac{1}{k^2\pi^2}\cosh\left(\dfrac{\pi^2}{4}\right)$ 　(d) $\dfrac{1}{2k\pi} e^{k\pi^2/4}$

Topic- Mathematical Physics

Sub topic-Integration

Ans.: (a)

Solution: $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \cosh kx^2 \sin^2 x\, dx \Rightarrow 2\int_{0}^{\frac{\pi}{2}} dx \cosh kx^2 \sin^2 x\, dx$ it is given k is very large so

$\cos kx^2 = \dfrac{\exp kx^2}{2}$ the integration

$$I = 2\int_{0}^{\frac{\pi}{2}} dx \frac{\exp kx^2}{2}\sin^2 x\, dx = \int_{0}^{\frac{\pi}{2}} dx\, 2kx \frac{\exp kx^2}{kx}\sin^2 x\, dx = \int_{0}^{\frac{\pi}{2}} dx\, 2kx \exp kx^2 \left(\frac{\sin^2 x}{2kx}\right) dx$$

Using integration by parts

$$I = \frac{\exp kx^2}{k}\left(\frac{\sin x}{x}\right)\sin x \Big|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} \frac{d}{dx}\left(\frac{\sin^2 x}{kx}\right)\exp kx^2 dx = \frac{1}{k\pi}\exp\left(\frac{k\pi^2}{4}\right) - 0$$

$\int_{0}^{\frac{\pi}{2}} \dfrac{d}{dx}\left(\dfrac{\sin^2 x}{kx}\right)\exp kx^2 dx$ will contain $\dfrac{1}{k^2}$ term which will be zero at large k.

So, $I = \dfrac{\exp\dfrac{\pi^2}{4}}{k\pi}$

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Q27. A hollow metal sphere filled with a thick, highly viscous oil is rotating about a vertical axis with an initial angular velocity ω_0. However, there is a small hole at the bottom of this sphere, through which drops of oil are leaking out vertically at a steady rate. The variation of the angular velocity $\omega(t)$ of the sphere with time t is best represented graphically by

(a) (b) (c) (d)
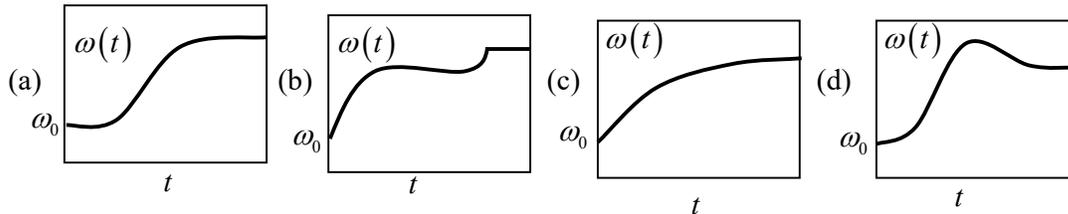
Topic-Mechanics

Sub topic- Fluid mechanics

Ans.: (a)

Q28. A pendulum which is suspended from the ceiling of a train has time period T_0 when the train is stationary. When the train moves with a small but steady speed v around a horizontal circular track of radius R, the time period of the pendulum will be

(a) $T_0\left(1-\dfrac{v^2T_0^2}{4\pi^2R}\right)^{-1/2}$ (b) $T_0\left(1+\dfrac{v^2T_0^2}{4\pi^2R}\right)^{-1/2}$ (c) $T_0\left(1-\dfrac{v^4}{g^2R^2}\right)^{-1/4}$ (d) $T_0\left(1+\dfrac{v^4}{g^2R^2}\right)^{1/4}$

Topic-Mechanics

Sub topic- Pendulum

Ans.(c)

$$\omega_0=\frac{2\pi}{T_0}=\sqrt{\frac{l}{g}}\Rightarrow \sqrt{\frac{l}{g}}=\frac{2\pi}{T_0}$$

The effective acceleration along the parallel to direction of tension is $g_{eff}=\sqrt{\left(\dfrac{v^2}{R}\right)^2+g^2}$

$$\omega=\frac{2\pi}{T}\Rightarrow T=\frac{2\pi}{\omega}$$

$$2\pi\sqrt{\frac{l}{g_{eff}}}=2\pi\sqrt{\frac{l}{\left(g^2+\left(\dfrac{v^2}{R}\right)^2\right)^{1/2}}}=2\pi\sqrt{\frac{l}{g\left(1+\dfrac{v^4}{R^2g^2}\right)^{1/2}}}=T\left(1+\frac{v^4}{R^2g^2}\right)^{-1/4}=T\left(1-\frac{v^4}{R^2g^2}\right)^{1/4}$$

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Q29. A cricket ball, bowled by a fast bowler, rises from the pitch at an angle of 30^0 with a speed of $72 km/hr$, then moves straight ahead and, at a height of $0.5m$, strikes the flat surface of the bat held firmly at rest in a horizontal position (see figure). As a result, the ball bounces off elastically, providing a return catch straight back to the bowler.
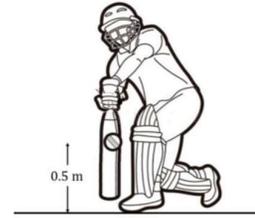
If the coefficient of restitution between the bat and the ball is 0.577, the acceleration due to gravity is $10ms^{-2}$ and air resistance can be neglected, the catch will carry, before hitting the ground, to a distance of approximately

(a) $19.5m$ (b) $37.0m$ (c) $9.5m$ (d) $21.0m$

Topic-Mechanics

Sub topic-Newtons law

Ans. (a)

Solution:

$20\cos 30 = 20\sqrt{3}/2 \rightarrow 10\sqrt{3}$

$V_y^2 = u_y^2 + 2as$

$V_y^2 = (10)^2 + 2 \times (-10)0.5$

$V_y^2 = 90$

$v_y = \sqrt{90} = 9.48$ m/s

After collision $\rightarrow$ ex $10\sqrt{3} = 0.577 \times 10\sqrt{3} = 9.99$ m/s

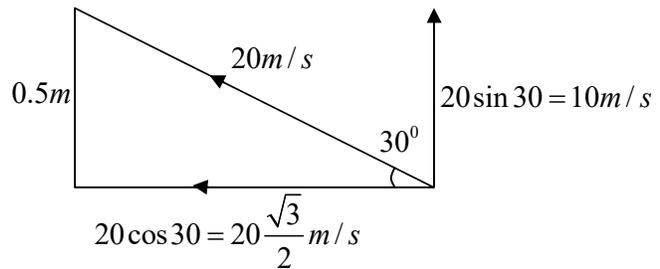After the collision, the y-component of velocity

$= 9 \cdot 48$ m/s

$s = ut + \dfrac{1}{2}gt^2$

$-0.5 = 9.48t + \dfrac{1}{2}(-10)t^{\wedge}2$

$5t^2 - 9.48t - 0.5 = 0$

$t = 1.94$

$s = ut = 9.99 \times 1.94 = 19.45$

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Q30. In a matrix mechanics formulation, a spin-1 particle has angular momentum components

$$L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & \sqrt{2} & 0 \\ -1 & 0 & -\sqrt{2} \end{pmatrix} \qquad L_z = \frac{\hbar}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

It follows that $L_y =$

(a) $\dfrac{\hbar}{2}\begin{pmatrix} 0 & -i & i \\ i & 0 & -i\sqrt{2} \\ -i & i\sqrt{2} & 0 \end{pmatrix}$

(b) $\dfrac{\hbar}{2}\begin{pmatrix} 0 & i & -i \\ -i & 0 & i\sqrt{2} \\ i & -i\sqrt{2} & 0 \end{pmatrix}$

(c) $\sqrt{2}\hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(d) $\sqrt{2}\hbar\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ -\sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

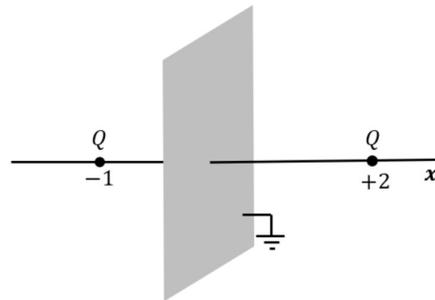**Topic-Quantum Mechanics**

Sub topic-Angular momentum

Ans. (a)

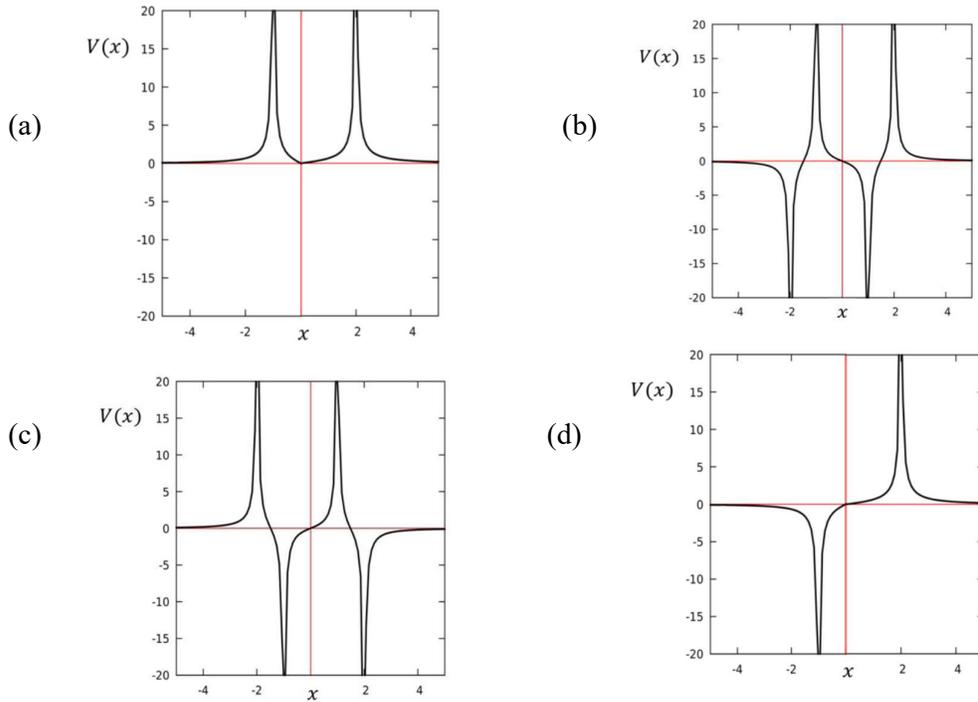Solution: The commutation between $[L_z, L_x] = i\hbar L_y \Rightarrow L_y = \dfrac{1}{i\hbar}(L_z L_x - L_x L_z)$

$$L_y = \frac{1}{i\hbar}\frac{\hbar^2}{4}\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & \sqrt{2} & 0 \\ -1 & 0 & -\sqrt{2} \end{pmatrix} - \begin{pmatrix} 0 & 1 & -1 \\ 1 & \sqrt{2} & 0 \\ -1 & 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\right)$$

$$L_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & i \\ i & 0 & -i\sqrt{2} \\ -i & i\sqrt{2} & 0 \end{pmatrix}$$

Q31. Two equal positive point charges $Q = +1$ are placed on either side of an x-axis normal to a grounded infinite conducting plane at distances of $x = +2$ units and $x = -1$ unit respectively (see figure) w.r.t. the point of intersection of the axis with the conducting plane as origin.

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The electrostatic potential along the axis will correspond to the graph in

(a)

(b)

(c)

(d)

Ans. (a)

Q32. Two co-axial solenoids A and B, one placed completely inside the other, have the following parameters:

Solenoid	Number of turns	Length	Diameter
A	1000	50cm	2cm
B	2000	50cm	4cm

The mutual inductance between the solenoids is

(a) $1.58\,mH$ (b) $125.7\,mH$ (c) $395.0mH$ (d) $12.57mH$

Ans.: (a)

Solution: $B = \mu_o ni = \dfrac{\mu_o N_A i}{l_A}$

$\phi = BA\cos o = \mu_o ni = \dfrac{\mu_o N_A i}{l_A}\dfrac{\pi d_A^2}{4}$

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Flux passes through solenoid B

$$N_B \phi == \frac{\mu_o N_B N_A i}{l_A} \frac{\pi d_A^2}{4}$$

The mutual inductance is

$$m = \frac{N_B \phi}{i} = \frac{\mu_o N_B N_A}{l_A} \frac{\pi d_A^2}{4}$$

$$m = \frac{\mu_o N_B N_A}{l_A} \frac{\pi d_A^2}{4} = \frac{4\pi \times 10^{-7} \times 1000 \times 2000 \times \pi \times (2 \times 10^{-2})^2}{4 \times 50 \times 10^{-2}}$$

$$m = 1.58 \, mH$$

Q33. A particle of mass m in a three-dimensional potential well has a Hamiltonian of the form

$$H = \frac{p_2^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} m\omega^2 y^2 + 2m\omega^2 z^2$$

Where ω is a constant. If there are two identical spin $-1/2$ particles in this potential having a total energy

$$E = 6\hbar\omega$$

The entropy of the system will be

(a) $k_B \ln 14$ (b) $k_B \ln 16$ (c) $k_B \ln 12$ (d) $k_B \ln 10$

Topic-Thermodynamics and statistical mechanics

Sub topic- Thermodynamics

Ans. (a)

Solution: $E = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar\omega + \left(n_z + \frac{1}{2}\right)2\hbar\omega$

$E = \left(n_x + n_y + 2n_z + 2\right)\hbar\omega$

$S = k_B \ln\Omega$

$n_x + n_y + 2n_z + 2 = 6$

$\Omega = 12, \quad S = k_B \ln 12$

1	2
$2\hbar\omega \rightarrow (0,0,0)$	$4\hbar\omega \rightarrow (0,0,1)$ $(1,1,0)$ $(2,0,0)$ $(0,0,2)$
$3\hbar\omega \rightarrow (1,0,0)$	$3\hbar\omega \rightarrow (1,0,0)$ $(0,1,0)$

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Q34. A quantum dot is constructed such that it has just three energy levels, with energies E, $2E$ and $3E$ respectively. The chemical potential in the system has the value $\mu = 2E$ and the temperature is given by

$$T = \frac{E}{2k_B}$$

The expected number of electrons populating the quantum dot will be

(a) 3.0   (b) 2.5   (c) 1.5   (d) 4.0

Topic: Thermodynamics and statistical mechanics

Sub topic: statistical mechanics

Ans.: (a)

Solution: The fermi Dirac distribution tells the probability of occupancy

$$f(E_i) = \frac{1}{e^{\frac{(E_i - \mu)}{k_B T}} + 1}$$

The chemical potential is given as $\mu = 2E \Rightarrow E = \frac{\mu}{2}$

$$f(E) = \frac{1}{e^{\frac{E_i}{k_B T}} e^{\frac{-\mu}{k_B T}} + 1} = \frac{1}{e^{\frac{E_i}{k_B T}} e^{\frac{-2E}{k_B T}} + 1} , \quad f(E_i) = \frac{1}{e^{\frac{E_i}{k_B T}} e^{\frac{-2E}{k_B T}} + 1}$$

At temperature $T = \frac{E}{2k_B}$

$$f(E_i) = \frac{1}{e^{\frac{2k_B E_i}{k_B E}} e^{\frac{-4k_B E}{k_B E}} + 1} = \frac{1}{e^{\frac{4E_i}{E}} e^{-4} + 1} , \qquad \text{At } E_i = E$$

$$f(E) = \frac{1}{e^{\frac{2E}{E}} e^{-4} + 1} = 0.88 , \qquad \text{At } E_i = 2E$$

$$f(2E) = \frac{1}{e^4 e^{-4} + 1} = 0.5 , \qquad \text{At } E_i = 3E$$

$$f(3E) = \frac{1}{e^6 e^{-4} + 1} = 0.27$$

The expected number of electrons

$$N = 2(0.88 + 0.5 + 0.27) = 3.3 \approx 3$$

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Q35. The non-inverting amplifier shown in the figure on the right is constructed using a non-ideal operational amplified (op amp) with a *finite* open loop gain A.
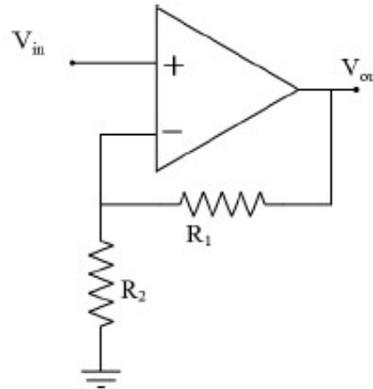
The value of feedback fraction is

$$B = \frac{R_2}{R_1 + R_2} = 0.1$$

If the gain A varies such that

$$10^4 < A < 10^5$$

Then the approximate percentage variation in the closed loop gain will be

(a) 0.09% (b) 0.0% (c) 0.9% (d) 9.0%

Topic-Electronics

Sub topic- Op-AmP

Ans. (a)

Solution : Closed loop gain

$$A_{CL} = \frac{A}{1 + AB}$$

If gain $A = 10^4$ and $B = 0.1$

Then

$$A_{CL} = \frac{10^4}{1 + 10^3} = 9.99$$

If gain $A = 10^5$ and $B = 0.1$

Then, $A_{CL} = \frac{10^5}{1 + 10^4} = 9.999$

Percentage variation in closed loop gain, $\Delta A_{CL}\% = \frac{9.999 - 9.99}{9.99} \times 100 = 0.09\%$

Q36. In a standardized entrance exam, the passing rates for the past 10 years are tabulated below.

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Passing Rate	22%	16%	23%	21%	22%	14%	17%	20%	24%	21%

If 1000 candidates appear for the exam every year, the probability that more than 250 students will pass the exam this year is about

(a) 6% (b) 20% (c) 25% (d) 0.1%

Topic-Mathematical Physics

Sub-topic- Probability

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Ans.: (a)

Solution: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$f(x)$ = probability density function
σ = standard deviation
μ = mean

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

Where:

- n is the number of observations (10 years).
- x_i is each individual passing rate.
- $\bar{x}$ is the mean passing rate.
- $\sigma = 3.28, \mu = 20$ can be calculated from the data.
- $\frac{\int_{25}^{100} e^{-(0.5(x-2\ )^2)/10.76} dx}{3.28\sqrt{2\pi}} = 0.0637245 \sim 6\ \%$

Q37. A spectrographic method to search for exoplanets is by measuring its velocity along the line of sight, using the Doppler shift in the spectrum. If a star of mass M and a planet of mass m are moving around their common centre of mass, this component of velocity will vary periodically with an amplitude.

$$A = \left(\frac{2\pi G_N}{T}\right)^{1/3} \frac{m}{M^{2/3}}$$

For a particular planet-star system, if the time period is $T = (12 \pm 0.3)$ years, and A and M are measured with an accuracy of 3% each, then the error in the measurement of the mass m is

(a) 3.7%   (b) 8.5%   (c) 5.8%   (d) 6.3%

Ans. (d)

Solution: The amplitude is given as
$A = (\frac{2\pi G_n}{T})^{1/3} \frac{m}{M^{2/3}}$

$A = (\frac{2\pi G_n}{T})^{1/3} \frac{m}{M^{2/3}} \Rightarrow m = \frac{AT^{1/3}}{(2\pi G_n)^{1/3}} M^{2/3} \Rightarrow \ln(m) = \ln(A) + \frac{2}{3}\ln(M) + \frac{1}{3}\ln(T) - \frac{1}{3}\ln(2\pi G_n)$

$\frac{\Delta m}{m} = \frac{\Delta A}{A} + \frac{2}{3}\frac{\Delta M}{M} + \frac{1}{3}\frac{\Delta T}{T},\ \frac{\Delta m}{m} \times 100 = 3\% + \frac{2}{3} 3\% + \frac{1}{3}\frac{0.3}{12} \times 100$

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$$\frac{\Delta m}{m} \times 100 = 3\% + \frac{2}{3} 3\% + \frac{1}{3} \frac{0.3}{12} \times 100 = 5.8\%$$

Q38. A satellite used to make Google Earth images carries on board a telescope which must be designed, when operating at a wavelength λ, to be able to resolve objects on the ground of length as small as δ.

If the satellite goes around the Earth in a circular orbit with uniform speed v, the minimum diameter

D_{min} of the telescope mirror can be determined in terms of R, the radius of the Earth, and

g, the acceleration due to gravity at the surface, to be

(a) $\dfrac{1.22\lambda}{\delta}\left(\dfrac{gR^2}{v^2} - R\right)$

(b) $\dfrac{1.22\lambda}{\delta} \dfrac{gR^2}{v^2}\left(1 + \dfrac{R}{\lambda}\right)$

(c) $\dfrac{1.22\lambda}{\delta} \dfrac{gR^2}{\lambda v^2}$

(d) $\dfrac{1.22\lambda}{\delta} \sqrt{\dfrac{gR^3}{v^2}}$
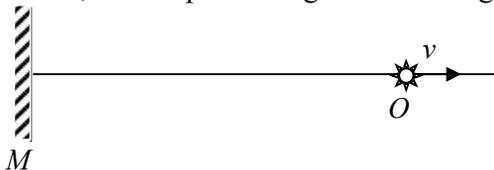
Topic-Mechanics

Sub topic- Gravity

Ans. (a)

Solution: $\dfrac{mv^2}{R+h} = \dfrac{GMm}{(R+h)^2} \Rightarrow (R+h) = \dfrac{GM}{v^2} \Rightarrow R = \dfrac{GM}{v^2} - h$

$D_{min} = \dfrac{1.22\lambda}{\Delta\theta} \quad \left(\Delta\theta = \dfrac{\delta}{h}\right)$

$D_{min} = \dfrac{1.22\lambda h}{\delta} = \dfrac{1.22\lambda}{\delta}\left(\dfrac{GM}{v^2} - R\right) = \dfrac{1.22}{\delta}\left(\dfrac{gR^2}{v^2} - R\right); \; GM = gR^2$

Q39. An observer O, moving with relativistic speed v away from a fixed plane mirror M in a line perpendicular to the mirror surface, sends a pulse of light of wavelength λ towards the mirror.

M

The wavelength of the light reflected back to the observer will be
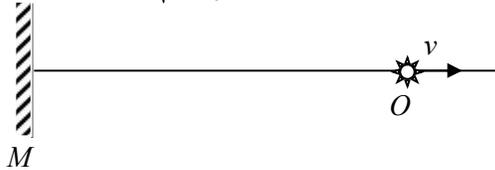
(a) $\lambda\left(\dfrac{c+v}{c-v}\right)$

(b) $\lambda\left(\dfrac{c+2v}{c-2v}\right)$

(c) $\lambda\left(\dfrac{c-v}{c+v}\right)$

(d) $\lambda\left(\dfrac{c-2v}{c+2v}\right)$

Topic-Oscillation, waves and optics

Sub topic-

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Ans.: (a)

Solution: Mirror receive a wavelength $\lambda_R = \lambda \sqrt{\dfrac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}}$ source (observer) is moving away from light

towards mirror fig.

Now observer receive the wave length $\lambda_0 = \lambda_R \sqrt{\dfrac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}} = \lambda \sqrt{\dfrac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}} \sqrt{\dfrac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}} = \dfrac{c+v}{c-v}$

(Again, observer is moving away from the light coming from mirror)

Q40. The low-temperature specific heat of a certain material is primarily due to acoustic phonons. The frequency ω of a phonon is related to its wavevector k by $\omega = ck$, where, c is the speed of sound in the material. The phonons have a Bose distribution

$$n(k) = \frac{1}{e^{hck/k_B T} - 1}$$

And the energy of a phonon has a maximum possible value ω_D

In a two-dimensional sample, the specific heat at low-temperature behaves as

(a) $\left(\dfrac{T}{\omega_D}\right)$ (b) $\left(\dfrac{T}{\omega_D}\right)^3$ (c) $\left(\dfrac{T}{\omega_D}\right)^{3/2}$ (d) $\dfrac{T}{\omega_D}$

Topic-Solid state Physics

Sub topic- Density of States

Ans.: (a)

Solution: The distribution of Phonon is given as follows

$$n(k) = \frac{1}{e^{\hbar c\omega/k_B T} - 1}$$

The density of states in 2-d can be written as follows

$$g(k)dk = 2\left(\frac{L}{2\pi}\right)^2 2\pi k\, dk \Rightarrow g(k)dk = \frac{L^2}{\pi} k\, dk$$

The wave vector can be written as follows

$$k = \frac{\omega}{c} \Rightarrow dk = \frac{d\omega}{c}$$

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$$g(\omega)d\omega = \frac{L^2}{\pi}\frac{\omega d\omega}{c^2} \Rightarrow N = \frac{L^2}{c^2\pi}\frac{\omega^2_D}{2} \Rightarrow \frac{N}{L^2} = \frac{\omega^2_D}{2\pi c^2} \Rightarrow \omega^2_D = 2\pi nc^2$$

The average energy can be written as follows

$$\langle E \rangle = \int_0^{\omega_D} \langle E \rangle_{eistein} dN = \int_0^{\omega_d} 2N\hbar\omega\left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right]\frac{L^2}{\pi c^2}\omega d\omega$$

$$= \int_0^{\omega_d} N\hbar\omega\frac{L^2}{\pi c^2}\omega d\omega + 2N\hbar\frac{L^2}{\pi c^2}\int_0^{\omega_d}\frac{1}{e^{-\beta\hbar\omega}-1}\omega d\omega = \left(A + \frac{B}{\theta_D^2}T^3\right)$$

As we know that the Debye temperature can be written as follows

$$\theta_d = \frac{\hbar\omega_d}{k_B}$$

At very low temperature

$$T < \theta_d$$

The average energy at very low temperature can be written as follows

$$\langle E \rangle \, \alpha \left(A + \frac{B}{\theta_D^2}T^3\right)$$

The heat capacity can be written as follows

$$C = \frac{d\langle E \rangle}{dT} = \frac{d}{dT}\left(A + \frac{B}{\theta_D^2}T^3\right) \Rightarrow C\,\alpha\,\frac{T^2}{\theta_D^2}$$

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Section C

(only for Ph.D. candidates)

This Section consists of 15 questions. All are of multiple-choice type. Mark only one option on the online interface provided to you. If more than one option is marked, it will be assumed that the question has not been attempted. A correct answer will get +5 marks, an incorrect answer will get 0 mark.

Q41. Consider the inner product in the space of normalisable functions defined on the interval $[-1,1]$

$$\langle F|g\rangle = \int_{-1}^{1} dx \left(1+x^2\right) f(x) g(x)$$

The projection of the vector 1 along the vector x^2 is

(a) $\dfrac{14}{9}x^2$ 　　　　(b) $\dfrac{16}{35}\sqrt{\dfrac{35}{24}}x^2$ 　　　　(c) $\dfrac{16}{15}x^2$ 　　　　(d) $\sqrt{\dfrac{35}{24}}x^2$

Topic-Quantum Mechanics

Sub topic- Vector space

Ans.: (a)

Solution: $\langle f|g\rangle = \int_{-1}^{1} \left(1+x^2\right) f(x) g(x)$

Here $f(x)=1, g(x)=x^2$

$$\langle f|f\rangle = \int_{-1}^{1}\left(1+x^2\right)1.1dx = 2\int_{0}^{1}\left(1+x^2\right)dx = 2\left(1+\frac{1}{3}\right) = \frac{8}{3}$$

Normalized $|f\rangle = \sqrt{\dfrac{3}{8}}$

$$\langle g|g\rangle = \int_{-1}^{1}\left(1+x^2\right)x^2.x^2dx = 2\left(\frac{1}{5}+\frac{1}{7}\right) = \frac{24}{35}$$

Normalised $|g\rangle = \sqrt{\dfrac{35}{24}}x^2$

Projection of unit vector $|f\rangle$ along unit vector $|g\rangle$ is

$$\langle f|g\rangle = \sqrt{\frac{3}{8}}.\sqrt{\frac{35}{24}}\int_{-1}^{1}\left(1+x^2\right)1.x^2dx = \sqrt{\frac{3}{8}}.\sqrt{\frac{35}{24}}2\int_{0}^{1}\left(x^2+x^4\right)dx = \sqrt{\frac{3}{8}}.\sqrt{\frac{35}{24}}2\left(\frac{1}{3}+\frac{1}{5}\right) = \sqrt{\frac{3}{8}}\sqrt{\frac{35}{24}}\frac{16}{15}$$

The projection is $\sqrt{\dfrac{3}{8}}\sqrt{\dfrac{35}{24}}\dfrac{16}{15}\sqrt{\dfrac{35}{24}}x^2 = \sqrt{\dfrac{3}{8}}\dfrac{35}{24}\times\dfrac{16}{15}x^2 = \sqrt{\dfrac{3}{8}}\dfrac{14}{9}x^2$

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The magnitude of $|f\rangle = \sqrt{\dfrac{8}{3}}$

So projection of $\sqrt{\dfrac{8}{3}}|f\rangle$ along direction $|g\rangle$ is $\sqrt{\dfrac{8}{3}} \times \sqrt{\dfrac{3}{8}} \dfrac{14}{9} x^2 = \dfrac{14}{9} x^2$

Q42. A dumbbell consists of two small spherical masses M each, connected by a thin massless rod of length 2ℓ.

This dumbbell is centred at the origin, and is rotating about the z - axis with a *uniform* angular velocity ω, making an angle θ with the z -axis (see figure).

Neglecting effects due to gravity, at the instant when the dumbbell is wholly in the yz -plane (as shown in the figure), the magnitude of torque about the origin will be
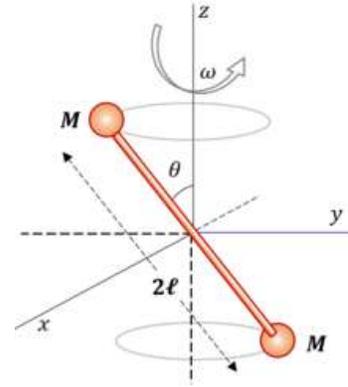
(a) $M\ell^2\omega^2 \sin 2\theta$ (b) $2M\ell^2\omega^2 \sin^2\theta$ (c) $2M\ell^2\omega^2 \cos^2\theta$ (d) Zero
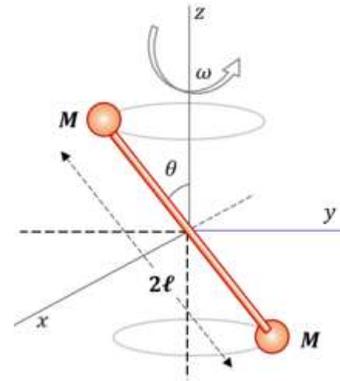
Topic-Mechanics

Sub topic- Moment of Inertia

Ans.: (a)

Solution: Moment of inertia about principal axis i.e. axis passing through mass less rod and center of mass is as origin is given by

$I_X = 2Ml^2, I_y = 0, I_z = 2Ml^2$ figure. $\vec{\omega} = (\omega\sin\theta, \omega\cos\theta, 0)$

$L = 2Ml^2\omega\sin\theta\,\hat{x}$

Torque $\vec{\omega} \times \vec{L} = 2Ml^2\omega^2 \sin\theta\cos\theta\,(\hat{y} \times \hat{x}) = Ml^2\omega^2 \sin 2\theta\,(-\hat{z})$

Q43. A system with two generalized coordinates (q_1, q_2) is described by the Lagrangian

$$L = m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \frac{3}{2}\dot{q}_2^2\right) - k\left(\frac{3}{2}q_1^2 + 2q_1q_2 + q_2^2\right)$$

where m is the mass, and k is a constant.

This system can execute oscillations with two possible time periods

(a) $T = 2\pi\sqrt{\dfrac{2m}{k}}$ and $T = 2\pi\sqrt{\dfrac{m}{2k}}$

(b) $T = 2\pi\sqrt{\dfrac{m}{2k}\left(5 - 2\sqrt{6}\right)}$ and $T = 2\pi\sqrt{\dfrac{m}{2k}\left(5 + 2\sqrt{6}\right)}$

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(c) $T = \pi\sqrt{\dfrac{m}{k}\left(1-\sqrt{15}\right)}$ and $T = \pi\sqrt{\dfrac{m}{k}\left(1+\sqrt{15}\right)}$

(d) $T = 2\pi\sqrt{\dfrac{2m}{3k}}$ and $T = 2\pi\sqrt{\dfrac{3m}{2k}}$

Topic-Classical Mechanics

Sub topic- Oscillation

Ans. (a)

Solution: $L = m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dfrac{3}{2}\dot{q}_2^2\right) - k\left(\dfrac{3}{2}q_1^2 + 2q_1q_2 + q_2^2\right)$

Kinetic energy $T = m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dfrac{3}{2}\dot{q}_2^2\right)$ in matrix form $T = \begin{pmatrix} 2m & 2m \\ 2m & 3m \end{pmatrix}$

Potential energy is $V = k\left(\dfrac{3}{2}q_1^2 + 2q_1q_2 + q_2^2\right)$ potential energy matrix is $\begin{pmatrix} 3k & 2k \\ 2k & 2k \end{pmatrix}$

Secular equation is $\left|V - \omega^2 T\right| = 0 \Rightarrow \begin{pmatrix} 3k - \omega^2 2m & 2k - 2\omega^2 m \\ 2k - 2\omega^2 m & 2k - \omega^2 3m \end{pmatrix} = 0$

$\left(3k - \omega^2 2m\right)\left(2k - \omega^2 3m\right) - \left(k - \omega^2 m\right)^2 = 0$

$6k^2 - 13k\omega^2 m + 6\omega^4 m^2 - k^2 + 2\omega^2 km - \omega^4 m^2 = 0$

$5k^2 - 11k\omega^2 m + 5\omega^4 m^2 = 0 \Rightarrow \omega^2 = \dfrac{11km \pm \sqrt{121k^2m^2 - 4\times 5 \times 5k^2m^2}}{10m^2}$

$\omega^2 = \dfrac{11k \pm \sqrt{11}k}{10m} \Rightarrow \omega_1 = \sqrt{\dfrac{k}{m}\left(\dfrac{11+\sqrt{11}}{10}\right)} = \omega_2 = \sqrt{\dfrac{k}{m}\left(\dfrac{11-\sqrt{11}}{10}\right)}$

$L = m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 + \dfrac{3}{2}\dot{q}_2^2\right) - k\left(\dfrac{3}{2}q_1^2 + 2q_1q_2 + q_2^2\right)$

$\left(3k - \omega^2 2m\right)\left(2k - \omega^2 3m\right) - \left(2k - 2\omega^2 m\right)^2 = 0$

$2k^2 - 5k\omega^2 m + \omega^4 m^2 = 0$

$k^2 - \dfrac{5}{2}k\omega^2 m + \omega^4 m^2 = 0 \Rightarrow \omega^4 - \dfrac{5}{2}\dfrac{k\omega^2}{m} + \dfrac{k^2}{m^2} \Rightarrow \omega^2 = \dfrac{\dfrac{5}{2}\dfrac{k}{m} \pm \sqrt{\dfrac{25}{4}\dfrac{k^2}{m^2} - 4\dfrac{k^2}{m^2}}}{2} = \dfrac{\dfrac{5}{2}\dfrac{k}{m} \pm \dfrac{3}{2}\dfrac{k}{m}}{2}$

$\omega = \dfrac{\dfrac{5}{2}\dfrac{k}{m} \pm \dfrac{3}{2}\dfrac{k}{m}}{2} = \dfrac{k}{2m}\left(\dfrac{5}{2} \pm \dfrac{3}{2}\right) = \dfrac{k}{2m}, \dfrac{2k}{m} \Rightarrow T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{2k}}, \; 2\pi\sqrt{\dfrac{2m}{k}}$

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Q44. A system was formed of three spin- $1/2$ particles A, B and C, respectively and it was prepared in an initial state

$$|\psi\rangle = c_1|\uparrow\uparrow\uparrow\rangle + c_2|\uparrow\uparrow\downarrow\rangle + c_3|\downarrow\uparrow\uparrow\rangle + c_4|\uparrow\downarrow\downarrow\rangle + c_5|\downarrow\uparrow\uparrow\rangle + c_6|\downarrow\uparrow\downarrow\rangle + c_7|\downarrow\downarrow\uparrow\rangle + c_8|\downarrow\downarrow\downarrow\rangle$$

where the symbols $|\uparrow\rangle$ and $|\downarrow\rangle$ indicate states with $S_z = +1/2$ (spin-up) and $S_z = -1/2$ (spin-down) respectively. A measurement was made on the system in the initial state and this identified the spin state of the particle A to be $|\downarrow\rangle$ (spin-down). Now the expectation value of $\langle S_z \rangle$ for the particle C could be calculated as

(a) $\dfrac{|c_5|^2 + |c_7|^2 - |c_6|^2 - |c_8|^2}{|c_5|^2 + |c_7|^2 + |c_6|^2 + |c_8|^2}$

(b) $\dfrac{c_5 + c_7 + c_6 - c_8}{|c_5|^2 + |c_7|^2 + |c_6|^2 + |c_8|^2}$

(c) $\dfrac{\left(c_5^* + c_7^* - c_6^* - c_8^*\right)\left(c_5 + c_7 - c_6 - c_8\right)}{|c_5|^2 + |c_7|^2 + |c_6|^2 + |c_8|^2}$

(d) $\dfrac{\left(c_5 + c_7\right)^*\left(c_5 + c_7\right) - \left(c_6 + c_8\right)\left(c_6 + c_8\right)}{|c_5|^2 + |c_7|^2 + |c_6|^2 + |c_8|^2}$

Topic-Mechanics

Sub topic- Spin

Ans.: (a)

Solution:

$$|\psi\rangle = c_1|\uparrow\uparrow\uparrow\rangle + c_2|\uparrow\uparrow\downarrow\rangle + c_3|\uparrow\downarrow\uparrow\rangle + c_4|\uparrow\downarrow\downarrow\rangle + c_5|\downarrow\uparrow\uparrow\rangle + c_6|\downarrow\uparrow\downarrow\rangle + c_7|\downarrow\downarrow\uparrow\rangle + c_8|\downarrow\downarrow\downarrow\rangle$$

If measurement have been done such that it is identified particle A has down state so the state will proportional to,

$$|\psi_{A\downarrow}\rangle = c_5|\downarrow\uparrow\uparrow\rangle + c_6|\downarrow\uparrow\downarrow\rangle + c_7|\downarrow\downarrow\uparrow\rangle + c_8|\downarrow\downarrow\downarrow\rangle$$

Now S_z is measured for C

The measurement is value $\dfrac{\hbar}{2}$ with probability $\dfrac{|c_5|^2 + |c_7|^2}{|c_5|^2 + |c_6|^2 + |c_7|^2 + |c_8|^2}$

And $-\dfrac{\hbar}{2}$ with probability $\dfrac{|c_6|^2 + |c_8|^2}{|c_5|^2 + |c_6|^2 + |c_7|^2 + |c_8|^2}$

$$\langle S_z \rangle = \left(\frac{\hbar}{2} \times \frac{|c_5|^2 + |c_7|^2}{|c_5|^2 + |c_6|^2 + |c_7|^2 + |c_8|^2}\right) + \left(-\frac{\hbar}{2} \times \frac{|c_6|^2 + |c_8|^2}{|c_5|^2 + |c_6|^2 + |c_7|^2 + |c_8|^2}\right)$$

$$\langle S_z \rangle = \frac{\hbar}{2}\left(\frac{|c_5|^2 + |c_7|^2 - |c_6|^2 - |c_8|^2}{|c_5|^2 + |c_6|^2 + |c_7|^2 + |c_8|^2}\right)$$

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Q45. A particle is confined to a one-dimensional lattice with a lattice spacing δ. In the position space, the Hamiltonian operator for this particle is given by the matrix

$$\mathcal{H} = E_0 \begin{pmatrix} \ddots & & & & & & & \\ & \cdots & \cdots & 0 & 0 & 0 & 0 \\ \cdots & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & & \ddots \end{pmatrix}$$

Noting that it commutes with the generator T of translations

$$\mathcal{T} = \begin{pmatrix} \ddots & & & & & & & \\ & \cdots & \cdots & 0 & 0 & 0 & 0 \\ \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots \end{pmatrix}$$

where $T = e^{i\mathcal{P}\delta/\hbar}$ in terms of the momentum operator $\mathcal{P}$, the energy of a state with momentum p will be

(a) $4E_0\sin^2(p\delta/2\hbar)$ (b) $E_0\cos(p\delta/\hbar)$ (c) $E_0\sin(p\delta/\hbar)$ (d) $E_0(p\delta/2\hbar)^2$

Topic- Solid state Physics

Sub topic- Band theory

Ans.: (a)

Solution: From the given Hamiltonian it is clear that the interaction energy of a particular site with its own is E_0 and the interaction energy with neighbour is $-E_0$.

The energy can be written as follows

$$\varepsilon = 2E_0 - 2E_0\left(e^{ik_x\delta} + e^{-ik_x\delta}\right) = 2E_0 - 2E_0\cos(k_x\delta) = 2E_0\left[1 - \cos(k_x\delta)\right] = 4_0\sin^2(\frac{k_x\delta}{2})$$

$$\varepsilon = 4E_0\sin^2(\frac{k_x\delta}{2}) = 4E_0\sin^2(\frac{p\delta}{2\hbar})$$

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Q46. According to a standard table, the refractive index of water at 4°C is 1.33 at a wavelength of 590 nm. However, a carefully performed experiment in the lab yielded a refractive index of 1.41.

Which one of the following statements could be the explanation of this discrepancy?

(a) The experiment was performed at a wavelength lower than 590 nm.

(b) The experiment was performed at a wavelength higher than 590 nm.

(c) The water sample was at a temperature lower than 4°C.

(d) The water sample was at a temperature much higher than 4°C.

Topic-Oscillation, waves and optics

Sub topic: optics

Ans.: (a)

Q47. The power radiated by a point charge q moving rapidly with a uniform speed v in a circle of radius R will be

(a) $\dfrac{q^2 c}{6\pi\varepsilon_0 R^2}\left(\dfrac{v^2}{c^2 - v^2}\right)^2$

(b) $\dfrac{q^2 c^2}{6\pi\varepsilon_0 R^2}\left(\dfrac{v^2}{c^2 - v^2}\right)^2$

(c) $\dfrac{q^3 c}{6\pi\varepsilon_0 R^4}\dfrac{v^2}{c^2 - v^2}$

(d) $\dfrac{q^3 c^3}{6\pi\varepsilon_0 R^3}\dfrac{v^2}{c^2 - v^2}$

Topic-EMT

Sub topic: Radiation

Ans.: (a)

Solution: Power radiated can be written as follows

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{v \times a}{c}\right|^2\right)$$

Here the angle is 90^0, $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

$$P = \frac{\mu_0 q^2 a^2}{6\pi c(1 - \frac{v^2}{c^2})^3}\left(1 - \left|\frac{v}{c}\right|^2\right) = \frac{q^2 a^2 c^4}{\varepsilon_0 6\pi c^3 (c^2 - v^2)^2} = \frac{q^2 v^4 c}{\varepsilon_0 6\pi R^2 (c^2 - v^2)^2}$$

$$p = \frac{q^2 c}{6\pi\epsilon_0 R^2}\left(\frac{v^2}{c^2 - v^2}\right)^2$$

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Q48. A pseudo-potential V_{12} between every pair of particles in an ideal gas is to be constructed which will reproduce the effects of quantum statistics if the gas particles are bosonic in nature. A correct formula for this, in terms of the inter-particle distance r_{12} and a mean distance λ, will be of the form

(a) $V_{12} = -k_B T \ln\left(1 + e^{-2\pi r_{12}^2/\lambda^2}\right)$

(b) $V_{12} = -k_B T \ln\left(1 - e^{-2\pi r_{12}^2/\lambda^2}\right)$

(b) $V_{12} = +k_B T \ln\left(1 + e^{-2\pi r_{12}^2/\lambda^2}\right)$

(d) $V_{12} = +k_B T \ln\left(1 - e^{-2\pi r_{12}^2/\lambda^2}\right)$

Topic-Thermodynamics and Statistical Mechanics

Sub topic: Statistical Mechanics

Ans.: (a)

Q49. Three students A, B and C are given identical counters and each is asked to measure the number of gamma rays emitted per second by a given radioactive source. They are expected to perform the counting many times and find the mean and the standard deviation. The students find the following:

Student	A	B	C
Measurement (counts/second)	482 ± 22	495 ± 10	501 ± 22

If a counting experiment conducted previously by the instructor on this same sample with another identical counter had recorded exactly 30,000 gamma rays in a minute, then which of the following interpretations is valid?

(a) The measurement by student B is too precise to be believable.

(b) The measurement of student B is more correct than that of student A.

(c) The measurements of A and C have too large standard deviations.

(d) The measurement of C is much more precise than that of A.

**Topic-Nuclear and particle Physics**

Sub topic: Radioactivity

Ans.: (a)

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Q50. A thyratron consists of a tube filled with Xenon gas which can be used as a high-power electrical switch. Electrons are emitted from a cathode K heated by a filament F, and made to accelerate to some energy E by a voltage V applied across the anode plate P.

Electrons that scatter from the Xe atoms get deviated from their path and hit the shield S, which is a conducting envelope that transports the electrons back to ground potential (see figure on the right). The rest of the electrons strike the plate and contribute to the plate current i_p.
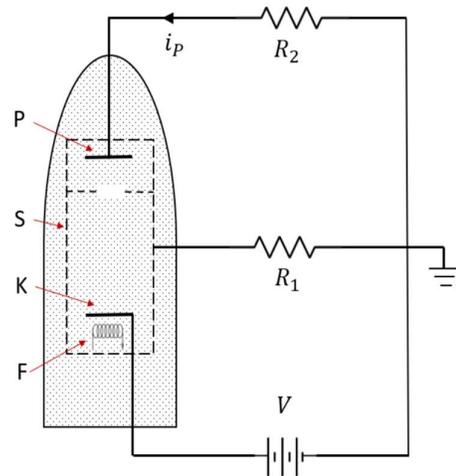
Which of the following graphs of the variation of the plate current i_p with increase in the accelerating voltage V could indicate the wave nature of the electron?

(a)

(b)

(c)

(d)
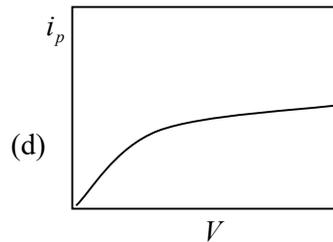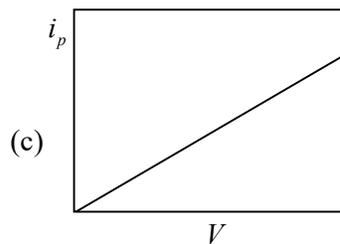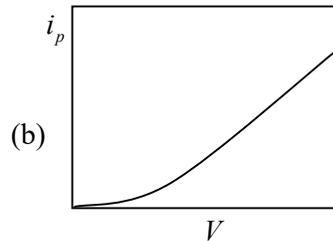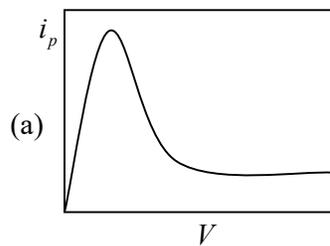
Topic-Quantum Mechanics

Sub topic- Scattering

Ans.: (a)

Solution: This problem can be solved from scattering amplitude

$$f(\theta,\phi) = (2l+1)e^{i\delta l}\sin(\delta_l)P_l(\cos\theta)$$

The total scattering correction can be written as, $\sigma_{tot} = |f(\theta,\phi)|^2$

At some particular applied potential,

At certain voltage we will have $\delta_l = \pi, 2\pi, \ldots\ldots$

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$f(\theta,\phi)=0$

$\sigma_{tot}=0$, it means no scattering. We will get maximum current.

So, the correct option will be (a)

Q51. In a semiclassical approach, the Hamiltonian of a He atom is modified by adding a magnetic interaction term between the two electrons, of the form

$$H_1 = A_2\vec{S}_1 \cdot \vec{S}_2$$

Where $\vec{S}_1$ and $\vec{S}_2$ are the electron spins and A_2 is a coupling constant. This leads, for the configuration $1s^2$, to the energy shift

(a) $-3A_2/4$ (b) $+3A_2/4$ (c) $+A_2/4$ (d) $-3A_2/4$

Topic-Atomic, Molecular and Laser Physics
Sub topic- Atomic Physis

Ans.: (a)

Solution: $H_1 = A_2\overrightarrow{S_1}.\overrightarrow{S_2}$

For two electrons system, net spin S can be written as follows

$$\vec{S}=\overrightarrow{S_1}+\overrightarrow{S_2} \Rightarrow \vec{S}.\vec{S}=\left(\overrightarrow{S_1}+\overrightarrow{S_2}\right).\left(\overrightarrow{S_1}+\overrightarrow{S_2}\right)=S_1^2+S_2^2+2S_1.S_2 \Rightarrow S_1.S_2 = \frac{S^2-S_1^2-S_2^2}{2}$$

$$H=A_2\overrightarrow{S_1}.\overrightarrow{S_2} \Rightarrow E=\langle H\rangle = A_2\left\langle\overrightarrow{S_1}.\overrightarrow{S_2}\right\rangle = A_2\frac{S(S+1)-S_1(S_1+1)-S_2(S_2+1)}{2}$$

Since, $\overrightarrow{S_1}$ and $\overrightarrow{S_2}$ both are $1/2$,

But, in $1s^2$ the electrons are anti parallel. Thus, $\vec{S}=0$

Thus, the energy shift can be written as follows, $E = \dfrac{0-\dfrac{3}{4}-\dfrac{3}{4}}{2}A_2 = -\dfrac{3}{4}A_2$

Q52. In the shell model of the nucleus, it is known that orbitals get filled in the order

$$1s_{1/2} \quad 1p_{3/2} \quad 1p_{1/2} \quad 1d_{5/2} \quad 2s_{1/2} \quad 1d_{3/2} \text{ and so on}$$

For a nucleus of $^{18}_{8}O$ the two neutrons outside the doubly-magic core of $^{16}_{8}O$ will occupy the same orbital. The allowed value of J^p will be

(a) 4^+ (b) 5^+ (c) 2^- (d) 3^+

Topic-Nuclear and Particle Physics

Sub topic- Shell Model

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Ans. (a)

Solution: Two neutrons are outside the core. They are at the same orbital. So the distribution will be

$$\left(1s_{1/2}\right)^2 \left(1p_{3/2}\right)^4 \left(1p_{1/2}\right)^2 \left(1d_{5/2}\right)^1 \left(1d_{3/2}\right)^1$$

Now the nuclear spin can be written as follows

$$J = \frac{5}{2} - \frac{3}{2} = 4$$

Q53. From the knowledge that you already have about the length of one year and the fact that the Sun subtends 0.5^0 in the sky, the average density of the Sun can be computed in $kg-m^{-3}$ as

(a) 1.7×10^3 (b) 7.5×10^3 (c) 1.7×10^2 (d) 7.5×10^2

Topic-Mechanics

Sub topic- Central force Problem

Ans.: (a)

Solution: $\dfrac{mv^2}{r} = \dfrac{GmM_s}{r^2} \Rightarrow (v)^2 = \dfrac{GM_s}{r}$

$v = \omega r = \dfrac{2\pi r}{T}$, $\left(\dfrac{2\pi r}{T}\right)^2 = \dfrac{GM_s}{r}$

$\Rightarrow r = \left(\dfrac{GM_s T^2}{4\pi^2}\right)^{1/3} = \left(\dfrac{6.67 \times 10^{-11} \times 1.987 \times 10^{30} (365 \times 24 \times 3600)^2}{4\pi^2}\right)^{1/3} \Rightarrow r = 14.9 \times 10^{11} \, m$

Now $D = \theta r = \dfrac{1.49 \times 10^{11} \times 0.5 \times \pi}{180} = 1.3 \times 10^9 \, m$; $D = $ Diameter of sun

The radius of sun can be written as, $R = D/2 = 6.5 \times 10^8 \, m$

The volume will be $V = \dfrac{4\pi}{3} R^3 = 1.16 \times 10^{27} \, m^3$

The density, $\rho = \dfrac{M_s}{V} = \dfrac{1.98 \times 10^{30}}{1.16 \times 10^{27}} = 1.7 \times 10^3$

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Q54. At very low temperatures, the electrical resistivity of most metals is dominated by

(a) Collisions of conduction electrons with impurity atoms and lattice vacancies.

(b) Absorption of conduction electrons by ions in the lattice.

(c) Collisions of conduction electrons with lattice phonons.

(d) Transfer of conduction electrons to the valence band.

Topic-Solid state Physics

Sub topic- Transport Properties

Ans.: (a)

Solution: At very low temperature the electrical resistivity of metal is dominated by collision conduction electrons with impurity atoms and impurity.

Q55. There are two conceivable channels by which a vector ρ^0 meson can decay into a pair of pseudoscalar pions. These are
$$\rho^0 \to \pi^0 + \pi^0 \text{ and } \rho^0 \to \pi^+ + \pi^-$$

The probability that the decay takes place through the process $\rho^0 \to \pi^+ + \pi^-$ is approximately

(a) 1   (b) $m_{\pi^0} / 2m_{\pi^+}$ (c) $m_{\pi^+}^2 / m_\rho^2$ (d) Zero

Topic-Nuclear and Particle Physics

Sub topic- Particle Physics

Ans. (a)

Solution: Note that this is just a general rule, and does not apply in 100% of cases. It *does* apply here for the primary decay- the probability for $\rho^0 \to \pi^+ + \pi^-$ is overwhelmingly higher than any of the others. But for instance, the decay $\rho^0 \to \pi^0 + \pi^0$ is just straight up forbidden by exchange symmetry- from conservation of angular momentum, the angular momentum of the final state implies that the state must be antisymmetric, but the fact that the final state is two identical bosons implies that the state must be symmetric. This is a contradiction, so the decay cannot occur at all. Thus, probability of $\rho^0 \to \pi^+ + \pi^-$ is 1.

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