Fermat Principle

Optical path:

When light travels a distance l in a medium of refractive index n the product nl is termed the optical path corresponding to the distance l. By definition,

$$n = \frac{c}{v}$$

Thus, the optical path $nl = \frac{c}{v} l = c \times$. Time taken by light to travel the distance l in the medium.

Fermat's Principle

The path followed by a ray of light in moving from one point to another point after any number of reflections or refractions would always be stationary (i.e., either maximum or minimum) with respect to variations of that path.

If the light from one point reaches another point directly or after reflection or refraction at a plane surface the path is minimum. Hence time taken by light in this case must also be minimum. Thus, the principle of least path may also be called the principal of least time. If the reflection or refraction occurs at spherical surfaces the path may be either maximum or-minimum, depending on the curvature of the surfaces. Hence, in general, the path would be stationary.

The optical path between two points may be expressed as $\sum nl \, or \int ndl$

From Fermat's principle,

$$\delta \sum nl = \delta \int ndl = 0 \ldots\ldots (1)$$

Rectilinear propagation of light:

If light moves from one point to another in a homogeneous medium, Fermat's principle implies that path of the ray between the two points should be least. Obviously, the path is a straight line. Thus, the rectilinear propagation of light results as direct consequence of Fermat's principle.

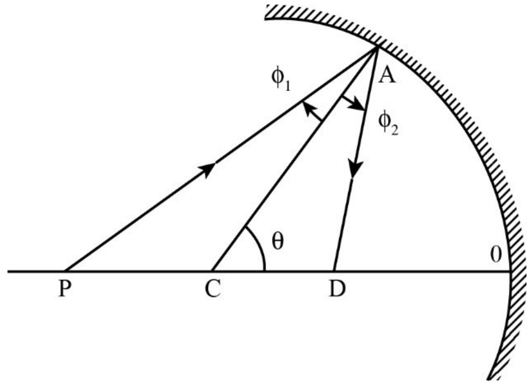
Principle of reversibility of light:

Fermat's principle stated in the form of the Equation (1), involves only the optical path and not the direction of propagation of light. SO a ray in going from the point P to Q will trace the same route as one from Q to P. Thus, the principle of reversibility is also included in Fermat's principle.

To establish laws of reflection at a spherical surface from Fermat's principle:

Suppose a ray of light from a fixed point P is arriving at another fixed point Q via the point A ofa splier ical surface as shown in Figure below. Let ϕ_1 and ϕ_2 be the angles which the rays PA and

AQ make with the normal CA at A drawn from the centre of curvature C of the spherical surface. Let $OC = R =$ radius of the spherical surface. Now from $\triangle APC$ and $\triangle AQC$ we can write



$\mathrm{PA} = (\mathrm{R}^2 + \mathrm{CP}^2 + 2\mathrm{R} \cdot \mathrm{CP} \cdot \cos\ \theta)^{\frac{1}{2}}$

$\mathrm{AQ} = (\mathrm{R}^2 + \mathrm{CQ}^2 - 2\mathrm{R} \cdot \mathrm{CQ} \cdot \cos\ \theta)^{\frac{1}{2}}$.......(2)

According to Fermat's principle the total optical path $\mathrm{PA} + \mathrm{AQ}$ between P and Q should be stationary and hence the variation of $\mathrm{PA} + \mathrm{AQ}$ with the angle 0 must be zero. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\mathrm{PA} + \mathrm{AQ}) = 0$$

Since R, CP and CQ are constants we get from above Equation

$$-\frac{CP}{PA} + \frac{CQ}{AQ} = 0 \Rightarrow -\frac{\sin\ \phi_1}{\sin\left(180^\circ - \theta\right)} + \frac{\sin\ \phi_2}{\sin\ \theta} = 0$$

or, $\phi_1 = \phi_2$ i.e, angle of incidence $(\phi_1) =$ angle of reflection (ϕ_2)

Reflection formula:

$$\cos\ \theta = 1 - \frac{\theta^2}{2}$$

Now assuming $\mathrm{OP} = \mathrm{x}$ and $\mathrm{OQ} = \mathrm{y}$ as the magnitude of distances we can write from equation (2)

$$\begin{aligned} PA \quad &= \left[R^2 + (x-R)^2 + 2R(x-R)\left(1 - \frac{\theta^2}{2}\right)\right]^{\frac{1}{2}} \\ &= [x^2 - R(x-R)\theta^2]^{\frac{1}{2}} = [x^2 - R(x-R)\theta^2]^{\frac{1}{2}} \\ &= x - \frac{1}{2}R^2\left(\frac{1}{R} - \frac{1}{x}\right)\theta^2 \ldots..(3) \end{aligned}$$

Similarly, we get

$$AQ \simeq y - \frac{1}{2}R^2\left(\frac{1}{R} - \frac{1}{y}\right)\theta^2 \ldots.(4)$$

Now for the optical path $L_{op} = PA + AQ$ to an extremum we must have $\frac{dL_{op}}{d\theta} = \frac{d}{d\theta}(PA + AQ) = 0$

Using Equation (3) and (4) we get, $\left[\frac{2}{R} - \frac{1}{x} - \frac{1}{y}\right]\theta = 0$ ……… (5)

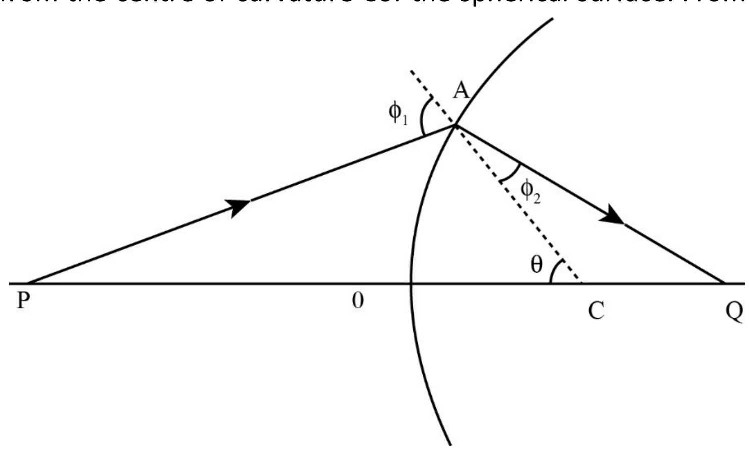
If the quantity within the bracket is not zero then we must have $\theta = 0$. In this case light falls normally on the surface along PO and gets reflected normally back to Q along OQ. Laws of reflection is obeyed here. On the other hand, if $\frac{1}{y} + \frac{1}{x} = \frac{2}{R}$ …. (6)

$\frac{dL_{op}}{d\theta}$ would vanish for all values of θ. This indicates that all paths like PAQ are allowed ray paths i.e., all paraxial rays emanating from P will, after reflection, pass through Q. Therefore, in this case Q will be the image point of the object point P Thus Equation (6) is reflection formula at the concave spherical surface. Using the sign convention that all the distances measured to the left of O are negative we may write object distance $OP = u = -x$, image distance $OQ = v = -y$ and radius of curyature $OC = r = -R$. Then formula (6) may be rewritten for a concave spherical reflector as

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$$

To establish laws of refraction at a spherical surface from Fermat's principle:

Suppose a ray of light from fixed point Q via the point A on a spherical surface separating two media of refractive indices n_1 and n_2, Let s_1 and $_2$ be the angles which the rays PA and AQ make with the normal CA at A drawn from the centre of curvature Cof the spherical surface. From $\triangle$ APC and $\triangle$ AQC we can write



$\mathrm{PA} = (\mathrm{AC}^{2} + \mathrm{CP}^2 - 2 \cdot \mathrm{AC} \cdot \mathrm{CP} \cdot \cos\ \theta)^{1/2}$ and $\mathrm{AQ} = (\mathrm{AC}^2 + \mathrm{CQ}^2 + 2 \cdot \mathrm{AC} \cdot \mathrm{CQ} \cdot \cos\ \theta)^{\overline{1/2}}$ …(7)

According to Fermat's principle the total optical path $n_1PA + n_2AQ$ between P and Q should be stationary and hence

$$\frac{d}{d\theta}(n_1PA + n_2AQ) = 0 \ldots (8)$$

Since AC, CP and CQ are constants, we get from Equation (8). Using

$\Rightarrow$ $n_1\sin\,\phi_1 = n_2\sin\,\phi_2$ which is the snell's law of refraction.

Refraction formula:

Under paraxial approximation the angle θ as indicated in (Figure above) is small so that we can use

$$\cos\,\theta = 1 - \frac{\theta^2}{2}$$

Now putting $OP = x, OQ = y$ and $AC = OC = R$ as the magnitude of the distances we can write from Equation (7),

$$PA = \left[R^2 + (x+R)^2 - 2R\cdot(x+R)\left(1 - \frac{\theta^2}{2}\right)\right]^{\frac{1}{2}}$$

$$= [x^2 + R(x+R)\theta^2]^{\frac{1}{2}} \simeq x\left[1 + \frac{R(x+R)\theta^2}{2x^2}\right] = x + \frac{1}{2}R^2\left(\frac{1}{x} + \frac{1}{R}\right)\theta^2$$

$$AQ \simeq y + \frac{1}{2}R^2\left(\frac{1}{R} - \frac{1}{y}\right)\theta^2$$

Now for the optical path $L_{0p} = n_1P \quad A + n_2AQ$ be an extremum we must have

$\frac{dL_{op}}{d\theta} = \frac{d}{d\theta}[n_1P\ A + n_2AQ] = 0, \Rightarrow \left[\frac{n_2}{y} + \frac{n_1}{x} - \frac{n_2 - n_1}{R}\right]\theta = 0$

$$\left[\frac{n_2}{y} + \frac{n_1}{x} - \frac{n_2 - n_1}{R}\right] = 0 \ldots\ldots (9)$$

$\frac{dL_{op}}{d\theta} = 0$ for all values of θ. This indicates that all paths like PAQ are allowed ray paths i.e., all paraxial rays emanating from P will, after refraction, pass through Q. Therefore, Q in that case will be the image point of the object point P. Equation (9) then represents the refraction formula at the curved surface.

Using the sign convention that all distances measured to the left of O are negative and those to its right positive we may write object distance $OP = u = -x$, image distance $OQ = v = y$ and radius of curvature $OC = r = R$. Then formula ... may be rewritten for a convex spherical surface in form

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r}$$