Energy of Harmonic Oscillator

The kinetic energy of system is given by $T=\frac{1}{2}m\dot{x}^2$ The potential energy of system is given by

$V=-\int Fdx=-\int -kxdx=\frac{1}{2}kx^2+c$ we have initial condition as $x=0,V=0$ so $c=0$

So potential energy is given by $V=\frac{1}{2}kx^2$

Total energy $E=T+V=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$ where $k=m\omega^2$ so $E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2$

$x=A\cos\left(\omega t+\delta\right)$ then the velocity is given by $\dot{x}=-A\omega\sin\left(\omega t+\delta\right)$

Total energy is given by $E=\frac{1}{2}m\omega^2A^2\sin^2\left(\omega t+\delta\right)+\frac{1}{2}m\omega^2A\cos^2\left(\omega t+\delta\right)=\frac{1}{2}m\omega^2A^2$

We can see total energy of harmonic oscillator is constant

So, one can write $\frac{1}{2}m\omega^2A^2=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2\Rightarrow\frac{1}{2}kA^2=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$

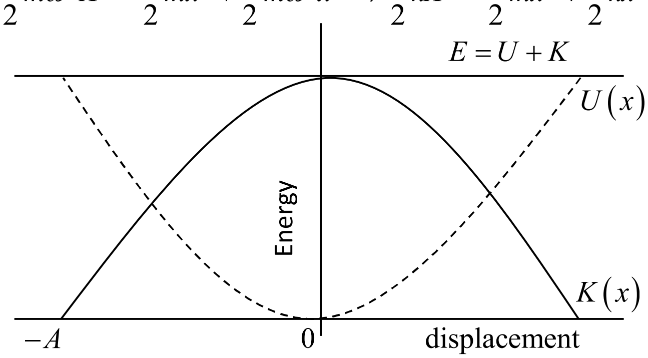
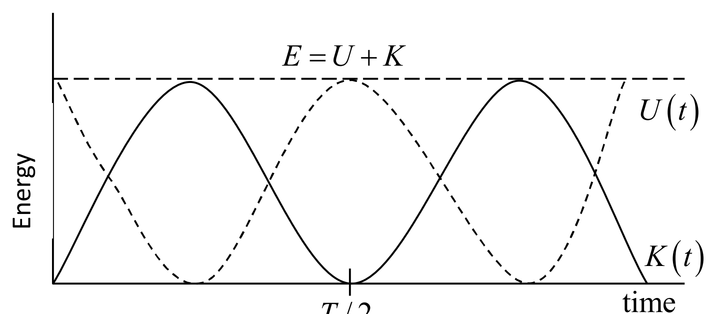


Figure: - Energy Diagram



Time average of potential energy

$$\langle V\rangle=\frac{\int_0^T V(t)dt}{\int_0^T dt}=\frac{\int_0^{T=\frac{2\pi}{\omega}}\frac{1}{2}kx^2dt}{\int_0^{T=\frac{2\pi}{\omega}}dt}=\frac{\frac{1}{2}kA^2\int_0^{T=\frac{2\pi}{\omega}}\cos^2(\omega t+\delta)dt}{\int_0^{T=\frac{2\pi}{\omega}}dt}=\frac{1}{4}kA^2$$

Time average of kinetic energy

$$\langle T\rangle=\frac{\int_0^T T(t)dt}{\int_0^T dt}=\frac{\int_0^{T=\frac{2\pi}{\omega}}\frac{1}{2}m\dot{x}^2dt}{\int_0^{T=\frac{2\pi}{\omega}}dt}=\frac{\frac{1}{2}m\omega^2A^2\int_0^{T=\frac{2\pi}{\omega}}\sin^2(\omega t+\delta)dt}{\int_0^{T=\frac{2\pi}{\omega}}dt}=\frac{1}{4}m\omega^2A^2=\frac{1}{2}kA^2$$

Time average of total energy

$$\langle E\rangle=\langle T\rangle+\langle V\rangle=\frac{1}{4}kA^2+\frac{1}{4}kA^2=\frac{1}{2}kA^2$$

The expression for power of damped harmonic oscillator can be written as follows

$$\langle E\rangle=\langle T\rangle+\langle V\rangle=\frac{1}{4}kA^2+\frac{1}{4}kA^2=\frac{1}{2}kA^2$$