Damped Oscillator

A particle of mass m execute simple Harmonic oscillation with a constant amplitude. if the resultant force on it is proportional to the displacement and directed opposite to it. But nature provides a large number of situations in which damping or resistance force will act on it then the amplitude and frequency will dependent on damping parameter.

The damping force are in general complicated in nature but we can completely solve the problem if damping force is linear in velocity

Let us assume damping force is $-bv$ where b is damping coefficient.

The equation of motion is given by

$$m\frac{d^2x}{dt^2} = -kx - bv \Rightarrow m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x + \frac{b}{m}\frac{dx}{dt}$$

Where $\omega_0^2 = \frac{k}{m}$ and $2\beta = \frac{b}{m}$ where ω_0 is natural frequency of oscillator without damping. Also, β is the damping parameter.

So, equation of motion can be represented as second order differential equation

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0 ....(1)$$

The general solution of equation of (1) is $x(t) = Ae^{\alpha t}$

$$Ae^{\alpha t}\left(\alpha^2 + 2\beta\alpha + \omega_0^2\right) = 0$$

$$\alpha = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$$

$$\alpha_1 = -\beta + \sqrt{\beta^2 - \omega_0^2},\ \alpha_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$$

$$x(t) = e^{-\beta t}\left(A_1 e^{(\sqrt{\beta^2 - \omega_0^2})t} + A_2 e^{(-\sqrt{\beta^2 - \omega_0^2})t}\right) ....(2)$$

Cases:

(1) $\beta = 0 \Rightarrow b = 0$, No drag force, (SHM)

(2) $\beta^2 < \omega_0^2 \Rightarrow b^2 < 4mk$, oscillation (Under damping)

(3) $\beta^2 > \omega_0^2 \Rightarrow b^2 > 4mk$, No oscillation (Over damping)

(4) $\beta^2 = \omega_0^2 \Rightarrow b^2 = 4mk$, No oscillation (Critical damping)

Case-1

$\beta = 0 \Rightarrow b = 0$, No drag force, Ideal SHM

Now substitute $\beta = 0$ in equation (2), we will get

$x(t) = C\cos\omega_0 t + D\sin\omega_0 t,\ where\, C = A_1 + A_2,\ D = i(A_1 - A_2)$ ....(3)

The equation (3) shows the SHM case where there is no damping.

Case -2

$\beta^2 < \omega_0{}^2 \Rightarrow b^2 < 4mk$, oscillation (Under damping)

Now let us define

$$\omega_1{}^2 = \omega_0{}^2 - \beta^2,\ A_1 = \frac{A}{2}e^{i\theta}\ \&\ A_1 = \frac{A}{2}e^{-i\theta}$$

Now, substituting these in equation (2), we will get

$$x(t) = Ae^{-\beta t}\cos\left(\omega_1 t + \theta\right).........(4)$$

$$A(t) = Ae^{-\beta t} = Amplitude \Rightarrow Decaying\ function,\ \cos\left(\omega_1 t + \theta\right) = Oscillating\ function$$

It is clearly noticeable from equation (4) that there is oscillation but the amplitude is exponentially decaying with time. The amplitude is decaying due to loss of energy caused by energy dissipation.

In under damping case,

We know that $A(t) = Ae^{-\beta t}$, Also, $E(t) = A(t)^2 = A^2 e^{-2\beta t} = E_0 e^{-2\beta t}$

Thus, energy is also decaying with time.

Relaxation time (τ) **:** The time in which energy stored in the system reduces to $1/e$ of its initial value is known as relaxation time.

$$E(t) = E_0 e^{-2\beta\tau} = \frac{E_0}{e}$$

$$e^{-2\beta\tau} = e^{-1} \Rightarrow \tau = \frac{1}{2\beta}$$

Rate of energy dissipation (P)/Power:

$$P = -\frac{dE}{dt} = 2\beta E(t) = \frac{b}{m}E(t) = \frac{E(t)}{\tau}$$

Quality factor (Q): $Q = 2\pi \times \dfrac{Energy\, stored}{Energy\, loss\ per\ cycle} = 2\pi \times \dfrac{E}{\frac{E}{\tau}T} = \omega_0 \tau = \omega_0 \dfrac{m}{b}$

Case -3 Over damping

$\beta^2 > {\omega_0}^2 \Rightarrow b^2 > 4mk$, No oscillation (Over damping)

Now let us define ${\omega_2}^2 = \beta^2 - {\omega_0}^2$

Now, substitute this in equation (2), we will get

$$x(t) = e^{-\beta t}\left(A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right) = \left(A_1 e^{-(\beta-\omega_2)t} + A_2 e^{-\beta+\omega_2)t}\right).....(5)$$

We can see that there is no oscillation. Also, the amplitude is decaying exponentially.

Case -4 Critical damping

$\beta^2 = {\omega_0}^2 \Rightarrow b^2 = 4mk$, Critical damping

Describe the trajectory under critical damping, let us start with original differential equation

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0$$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \beta^2 x = 0 \left[\sin ce, \beta = \omega_0\ \right]$$

$$\left(\frac{d}{dt} + \beta\right)\left(\frac{d}{dt} + \beta\right)x = 0, \quad let \left(\frac{d}{dt} + \beta\right)x = y$$

$$\left(\frac{d}{dt} + \beta\right)y = 0$$

$$\int \frac{dy}{y} = \int -\beta dt \Rightarrow \ln y = -\beta t \Rightarrow y = Ae^{-\beta t}\left[A = cons\tan t\right]$$

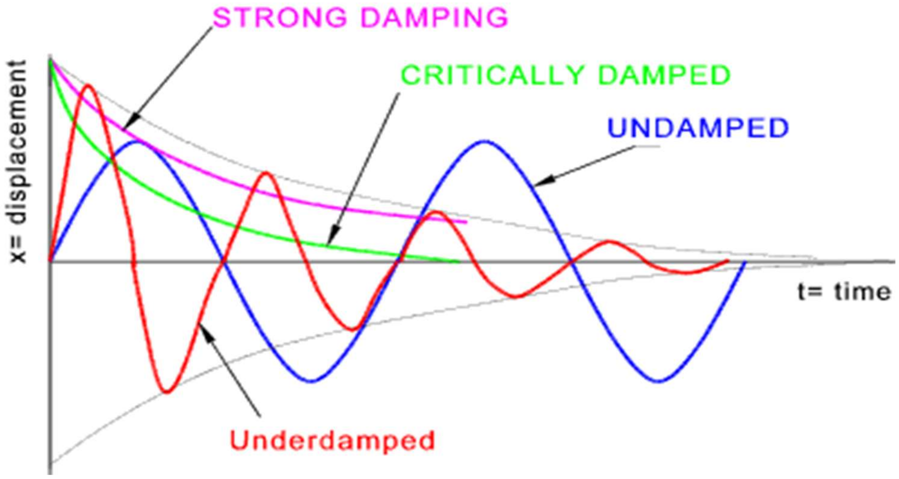
$$\left(\frac{d}{dt} + \beta\right)x = Ae^{-\beta t}$$

$$\left(\frac{d}{dt} + \beta\right)xe^{\beta t} = A$$

$$\left(\frac{d}{dt}xe^{\beta t}\right) = A$$

$$x = (At + B)e^{-\beta t}$$

Since, the exponential term is dominating. Thus, there is a rapid decaying in amplitude. Also, there is no oscillation under critical damping.


STRONG DAMPING
CRITICALLY DAMPED
UNDAMPED
x= displacement
t= time
Underdamped