Forced Oscillation

In this case, the oscillation is driven by some oscillating force

Let us consider $F_0 \sin(pt)$ is the external force which is applied to drive the oscillation. Here, p is representing the driving frequency

Now the equation of motion under the presence of restoring force, damping force and external force is

$$F_{net} = -kx - b\frac{dx}{dt} + F_0 \sin(pt)$$

$-b\frac{dx}{dt}$ =Damping force

Where, $-kx$ =restoring force

$F_0 \sin(pt)$=External periodic force

$$ma = -kx - b\frac{dx}{dt} - F_0 \sin(pt)$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin(pt)$$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = f_0 \sin(pt).....(1) \qquad 2\beta = b/m,\ \omega_0^2 = k/m,\ f_0 = F_0/m$$

The general solution of equation (1) is $x = A\sin(pt-\theta)$, Substitute this in equation (1), we will get

$$-p^2 A\sin(pt-\theta) + 2\beta pA\cos(pt-\theta) + \omega_0^2 A\sin(pt-\theta) = f_0 \sin(pt-\theta+\theta)$$

$$-p^2 A\sin(pt-\theta) + 2\beta pA\cos(pt-\theta) + \omega_0^2 A\sin(pt-\theta)$$

$$= f_0\left(\sin(pt-\theta)\cos\theta + \cos(pt-\theta)\sin\theta\right)......(2)$$

Now, comparing the coefficient of $\sin\theta$ and $\cos\theta$, we will get

$$\left(\omega_0^2 - p^2\right)A = f_0\cos(\theta)......(3)$$

$$2p\beta A = f_0 \sin(\theta)........(4)$$

$$\tan(\theta) = \frac{2p\beta}{\left(\omega_0^2 - p^2\right)}$$

From equation (3) and (4), we can write

$$A=\frac{f_0}{\sqrt{\left(\omega_0{}^2-p^2\right)^2+4\beta^2p^2}}......(5)$$

$$x(t)=\frac{f_0}{\sqrt{\left(\omega_0{}^2-p^2\right)^2+4\beta^2p^2}}\sin\left(pt-\theta\right).....(6)$$

Case 1

At very low driving frequency, $p<<\omega_0$

We will get the amplitude from equation (5)

$$A=\frac{f_0}{\omega^2}=\frac{F_0}{K}$$

Thus, A depends on force constant A but independent on driving frequency.

$$x(t)=\frac{F_0}{K}\sin\left(pt-\theta\right)$$

Case 2

At very high driving frequency, $p>>\omega_0$

$$A=\frac{f_0}{p^2}=\frac{F_0}{mp^2}$$

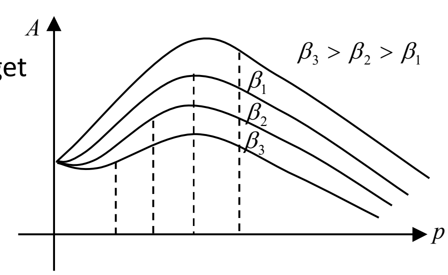
Thus, A depends on driving frequency.

Amplitude Resonance:

The value of driving frequency p at which amplitude become maximum is called amplitude resonance.

From, equation (5), it is clearly noticeable that $A=A_{\max}$, when

$$\frac{d}{dp}\left[\left(\omega_0{}^2-p^2\right)^2+4\beta^2p^2\right]=0\Rightarrow-4\omega_0{}^2+4p^2+8\beta^2=0\Rightarrow p=\sqrt{\omega_0{}^2-2\beta^2}$$



Now, if we substitute $p=\sqrt{\omega_0{}^2-2\beta^2}$ in equation (5), we will get

$$A_{\max}=\frac{f_0}{\sqrt{4\beta^4+4\beta^2\left(\omega^2{}_0-2\beta^2\right)}}$$

$$A_{\max}=\frac{f_0}{\sqrt{4\beta^2\omega^2{}_0-4\beta^4}}=\frac{f_0}{2\beta\sqrt{p^2+\beta^2}}$$

Velocity

$$x(t)=\frac{f_0}{\sqrt{\left(\omega_0^2-p^2\right)^2+4\beta^2p^2}}\sin\left(pt-\theta\right)$$

The expression for velocity can be written as,

$$v\left(t\right)=\frac{dx(t)}{dx}=\frac{f_0p}{\sqrt{\left(\omega_0^2-p^2\right)^2+4\beta^2p^2}}\cos\left(pt-\theta\right)=v_A\cos\left(pt-\theta\right)$$

$$v_A=\frac{f_0p}{\sqrt{\left(\omega_0^2-p^2\right)^2+4\beta^2p^2}}$$

Case 1

At very low driving frequency, $p<<\omega_0$

$$v_A=\frac{f_0p}{\omega_0^2}=\frac{F_0p}{K}$$

Case 2

At very high driving frequency, $p>>\omega_0$

$$v_A=\frac{f_0p}{p^2}=\frac{F_0}{mp}$$

Expression for maximum velocity

$$v_A=\frac{f_0}{\sqrt{\frac{\left(\omega_0^2-p^2\right)^2}{p^2}+4\beta^2}}$$

$$v_A=v_A^{\max},\ when\ p=\omega_0,\ v_A^{\max}=\frac{f_0}{2\beta}$$

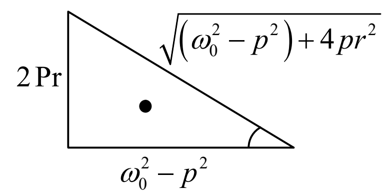
Power dissipated by oscillator

$$P=-F_dv=b\left(\frac{dx}{dt}\right)^2=\frac{bf_0^2p^2}{\left(\omega_0^2-p^2\right)^2+4\beta^2p^2}\cos^2\left(pt-\theta\right)$$

The average power will be, $P_{av}=\frac{bf_0^2p^2}{\left(\omega_0^2-p^2\right)^2+4\beta^2p^2}\left\langle\cos^2\left(pt-\theta\right)\right\rangle$

$$=\frac{1}{2}\frac{bf_0^2p^2}{\left(\omega_0^2-p^2\right)^2+4\beta^2p^2}=\frac{m\beta f_0^2}{\frac{\left(\omega_0^2-p^2\right)^2}{p^2}+4\beta^2}$$

$$P_{av}^{\max} = \frac{mf_0^{\ 2}}{4\beta}\left(\omega = p_0\right)$$



Power absorbed by oscillator

$$P = F_0 \sin(pt) v = \frac{F_0 f_0 p}{\sqrt{\left(\omega_0^{\ 2} - p^2\right)^2 + 4\beta^2 p^2}} \sin\left(pt\right)\cos\left(pt - \theta\right)$$

$$= \frac{F_0 f_0 p}{\sqrt{\left(\omega_0^{\ 2} - p^2\right)^2 + 4\beta^2 p^2}}\left[\frac{\sin\left(2pt\right)}{2}\cos\left(\theta\right) + \sin^2\left(pt\right)\sin\left(\theta\right)\right]$$

The average power will be

$$P_{av} = \frac{F_0 f_0 \beta p^2}{\left(\omega_0^{\ 2} - p^2\right)^2 + 4\beta^2 p^2} = \frac{F_0 f_0 \beta}{\frac{\left(\omega_0^{\ 2} - p^2\right)^2}{p^2} + 4\beta^2}$$

The maximum average power will be, $P_{av}^{\max} = {mf_0^{\ 2}}/{4\beta} \qquad \left(\omega = p_0\right)$

Band width of resonance

The difference in values of driving frequency at which the power absorbed drops to half of its maximum value is known as band width resonance.

$$\frac{F_0 f_0 \beta p^2}{\left(\omega_0^{\ 2} - p^2\right)^2 + 4\beta^2 p^2} = {mf_0^{\ 2}}/{8\beta} \Rightarrow \left(\omega_0^{\ 2} - p^2\right)^2 + 4\beta^2 p^2 = 8\beta^2 p^2$$

$$\Rightarrow \left(\omega_0^{\ 2} - p^2\right) = \pm 2\beta\ p$$

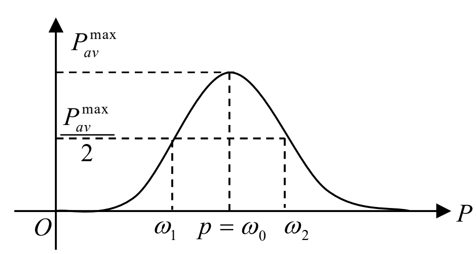
$$\Rightarrow \left(\omega_0 - p\right) = \pm\frac{2\beta\ p}{\left(\omega_0 + p\right)} = \pm\frac{2\beta}{\left(\frac{\omega_0}{p} + 1\right)} = \pm\frac{2\beta}{\left(1+1\right)} = \pm\beta \quad \left(\omega_0 \approx p\right)$$

$$\Rightarrow \left(\omega_0 - \omega_1\right) = \beta$$

$$\Rightarrow \left(\omega_0 - \omega_2\right) = -\beta$$

$$\Rightarrow \left(\omega_0 - \omega_1\right) = \beta$$

$$\Rightarrow \left(\omega_2 - \omega_1\right) = 2\beta = \frac{1}{\tau}$$



Quality factor (Q):

$$Q = \frac{1}{2}\left[1 + \left(\frac{\omega_0}{p}\right)^2\right]\left(p\tau\right)$$