Phase Velocity

The phase velocity of a wave is the rate at which the wave propagates in any medium. This is the velocity at which the phase of any one frequency component of the wave travels. For such a component, any given phase of the wave (for example, the crest) will appear to travel at the phase velocity.

We know that the wave equation can be written as

$$y = A\sin(\omega t - kx)$$

The phase velocity is defined as

$$v_p = \frac{\omega}{k}$$

In terms of wave length (λ) and time period (T) the phase velocity can be written as

$$v_p = \frac{\lambda}{T}$$

Group Velocity

The velocity with which the maximum amplitude of the wave group propagates in the medium is known as group velocity.

$$y_1(t) = a\sin(\omega_1 t - k_1 x)$$

$$y_2(t) = a\sin(\omega_2 t - k_2 x)$$

$$y = y_1(t) + y_2(t)$$

$$= 2a\sin\left(\left(\frac{\omega_1 - \omega_2}{2}t\right) - \left(\frac{k_1 - k_2}{2}x\right)\right) \times \cos\left(\left(\frac{\omega_1 + \omega_2}{2}t\right) - \left(\frac{k_1 + k_2}{2}x\right)\right)$$

$$v_g = \frac{\frac{\omega_1 - \omega_2}{2}}{\frac{k_1 - k_2}{2}} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{d\omega}{dk}$$

Relation between group velocity and phase velocity

$$v_g = \frac{d(kv_p)}{dk} = v_p + k\frac{d(v_p)}{dk}, \quad k = \frac{2\pi}{\lambda} \rightarrow dk = -\frac{2\pi}{\lambda^2}d\lambda$$

$$v_g = v_p - \lambda\frac{d(v_p)}{d\lambda}$$

Case: 1

If $\frac{d(v_p)}{d\lambda} = 0$, non-dispersive medium

$v_g = v_p$

Example: $\omega = ck$; $v_p = \frac{\omega}{k} = c$ and $v_g = \frac{d\omega}{dk} = c$

Case: 2

If $\frac{d(v_p)}{d\lambda} = +Ve$ Then it is called Normal dispersive medium

$v_g < v_p$

Example: Propagation of em wave in plasma

$\omega^2 = c^2k^2 + \omega_p{}^2$; $\omega_p =$ Plasma frequency; $\omega_p = \sqrt{\frac{n_0 e^2}{m\varepsilon_0}}$

$$\frac{\omega^2}{k^2} = c^2 + \frac{\omega_p{}^2}{k^2}$$

$$\frac{\omega}{k} = v_p = \sqrt{c^2 + \frac{\omega_p{}^2}{k^2}} > c$$

$$2\omega\frac{d\omega}{dk} = 2c^2k \Rightarrow v_g = \frac{c^2}{v_p} = \frac{c^2}{\sqrt{c^2 + \frac{\omega_p{}^2}{k^2}}} < c$$

Case: 3

If $\frac{d(v_p)}{d\lambda} = -Ve$ Then it is called Anomalous dispersive medium

$v_g > v_p$