Superposition of SHMs

Case-1 Addition of two SHMs having same frequency but different amplitude.

In this case we can write the SHMs equations as

$$y_1(t) = a_1 \sin(\omega t + \delta_1) \;\; \& \; y_2(t) = a_2 \sin(\omega t + \delta_2)$$

The resultant amplitude can be written as

$$y(t) = y_1(t) + y_2(t) = a_1 \sin(\omega t + \delta_1) + a_2 \sin(\omega t + \delta_2)$$

$$y(t) = \sin(\omega t)\{a_1 \cos(\delta_1) + a_2 \cos(\delta_2)\} + \cos(\omega t)\{a_1 \sin(\delta_1) + a_2 \sin(\delta_2)\} ....(1)$$

Let us assume,

$$\{a_1 \cos(\delta_1) + a_2 \cos(\delta_2)\} + \cos(\omega t)\{a_1 \sin(\delta_1) + a_2 \sin(\delta_2)\}$$

$$a_1 \cos(\delta_1) + a_2 \cos(\delta_2) = A\cos(\theta) ......(2)$$

$$a_1 \sin(\delta_1) + a_2 \sin(\delta_2) = A\sin(\theta) ......(3)$$

Now, substitute equation (2) and (3) in (1), we will get

$$y(t) = A\sin(\omega t + \theta) ....(4)$$

It is clearly noticeable that the resultant wave is SHM.

The amplitude A can be calculated from equations (2) and (3)

$$A = \sqrt{\left(a_1^{\,2} + a_2^{\,2} + 2a_1 a_2 \cos(\delta_1 - \delta_2)\right)}$$

$$A_{\max} = a_1 + a_2, \; when \; \cos(\delta_1 - \delta_2) = 2n\pi, n = 0,1,2,3....$$

$$A_{\min} = a_1 - a_2, \; when \; \cos(\delta_1 - \delta_2) = (2n+1)\pi, n = 0,1,2,3....$$

$$\tan(\theta) = \frac{a_1 \sin(\delta_1) + a_2 \sin(\delta_2)}{a_1 \cos(\delta_1) + a_2 \cos(\delta_2)}$$

Case-2 Addition of two SHMs having same amplitude but different frequency

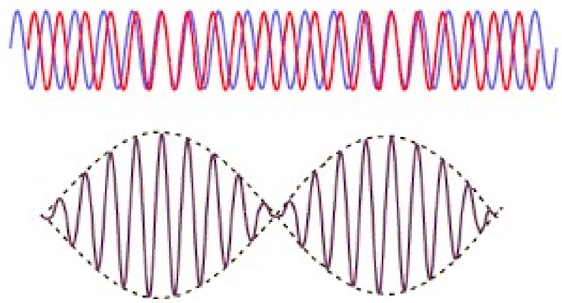
$$y_1(t) = a\sin(\omega_1 t) \;\; \& \; y_2(t) = a\sin(\omega_2 t)$$

The resultant wave can be written as

$$y(t) = y_1(t) + y_2(t) = a\sin(\omega_1 t) + a\sin(\omega_2 t) = 2a\cos\frac{(\omega_1 - \omega_2)}{2}t \sin\frac{(\omega_1 + \omega_2)}{2}t$$

$$y(t) = y_1(t) + y_2(t) = a\sin(\omega_1 t) + a\sin(\omega_2 t) = 2a\cos\frac{(\omega_1 - \omega_2)}{2}t \sin\frac{(\omega_1 + \omega_2)}{2}t$$

$y(t) = A(t)\sin\frac{(\omega_1 + \omega_2)}{2}t \Rightarrow$ Resultant wave is periodic but not SHM



Where $A(t) = 2a\cos\frac{(\omega_1 - \omega_2)}{2}t$ =Amplitude

We know that the intensity, $I\alpha\ Amplitude^2$

Thus, $I = 4a^2\cos^2\frac{(\omega_1 - \omega_2)}{2}t$

Intensity maximum

$$\cos^2\frac{(\omega_1 - \omega_2)}{2}t = 1 \Rightarrow \cos\frac{(\omega_1 - \omega_2)}{2}t = \cos n\pi$$

$$\frac{(\omega_1 - \omega_2)}{2}t = n\pi,\ n = 0,1,2,3$$

$$\frac{(2\pi n_1 - 2\pi n_2)}{2}t = n\pi,\ n = 0,1,2,3$$

$$t = 0, \frac{1}{n_1 - n_2}, \frac{2}{n_1 - n_2}........$$

Time difference between to maximum intensity $\Delta t = \frac{1}{n_1 - n_2}$

Thus, the beat frequency $f_b = n_1 - n_2$

Intensity minima

$$\cos^2\frac{(\omega_1 - \omega_2)}{2}t = 0 \Rightarrow \cos\frac{(\omega_1 - \omega_2)}{2}t = \cos(2n+1)\frac{\pi}{2}$$

$$\frac{(2\pi n_1 - 2\pi n_2)}{2}t = (2n+1)\frac{\pi}{2},\ n = 0,1,2,3$$

$$t = \frac{1}{2}\frac{1}{n_1 - n_2},\ \ \frac{3}{2}\frac{1}{n_1 - n_2},........$$

Time difference between to minimum intensity $\Delta t = \frac{1}{n_1 - n_2}$

Thus, the beat frequency $f_b = n_1 - n_2$

Case-3 Addition of two SHMs having different amplitude but different frequency

$$y_1(t) = a_1\sin(\omega_1 t)\ \&\ y_2(t) = a_2\sin(\omega_2 t)$$

$$y_1(t) = a_1\sin\left(\frac{\omega_1 + \omega_2}{2} + \frac{\omega_1 - \omega_2}{2}\right)t,\quad y_2(t) = a_2\sin\left(\frac{\omega_1 + \omega_2}{2} - \frac{\omega_1 - \omega_2}{2}\right)t,$$

Let, $\frac{\omega_1 + \omega_2}{2} = \omega_{av} \Rightarrow \frac{\omega_1 - \omega_2}{2} = \omega_m$

$y_1(t) = a_1\sin(\omega_{av}+\omega_m)t,\quad y_2(t) = a_2\sin(\omega_{av}-\omega_m)t,$

The resultant wave can be written as follows

$$y(t) = y_1(t)+y_2(t) = a_1\sin(\omega_{av}+\omega_m)t + a_2\sin(\omega_{av}-\omega_m)t$$

$$= \sin\omega_{av}t\left(a_1\cos\omega_m t + a_2\cos\omega_m t\right) + \cos\omega_{av}t\left(a_1\sin\omega_m t + a_2\sin\omega_m t\right) = A\sin(\omega_{av}t+\theta)$$

Where, $A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos(2\omega_m t)},\ \tan\theta = \left(\dfrac{a_1-a_2}{a_1+a_2}\right)\tan(\omega_m t)$

Case-4 Addition of N number of SHMs having same amplitude but different frequency

$y_1(t) = a\cos(\omega t),\ \ y_2(t) = a\cos(\omega+\delta\omega)t, y_2(t) = a\cos(\omega+2\delta\omega)t\ldots\ldots y_N(t) = a\cos(\omega_2)t$

We can write

$$\delta\omega = \frac{\omega_2-\omega_1}{N-1}$$

The resultant amplitude can be written as follows

$$y(t) = y_1(t)+y_2(t)+y_3(t)\ldots\ldots+y_N(t)$$

$$= a\cos(\omega t) + a\cos(\omega+\delta\omega)t + a\cos(\omega+2\delta\omega)t\ldots\ldots + a\cos(\omega_2)t$$

$$= a\,\mathrm{Re}\left[e^{i\omega_1 t}\{1+e^{i\delta\omega t}+e^{i2\delta\omega t}+\ldots\ldots\ldots\ldots+e^{i(N-1)\delta\omega t}\}\right]$$

$$= a\,\mathrm{Re}\left[e^{i\omega_1 t}\frac{e^{\frac{iN\delta\omega t}{2}}}{e^{\frac{i\delta\omega t}{2}}}\frac{\{e^{\frac{iN\delta\omega t}{2}}-e^{\frac{-iN\delta\omega t}{2}}\}}{\{e^{\frac{i\delta\omega t}{2}}-e^{\frac{-i\delta\omega t}{2}}\}}\right] = a\frac{\sin(\frac{N\delta\omega t}{2})}{\sin(\frac{\delta\omega t}{2})}\cos\frac{(\omega_1+\omega_2)}{2}t$$

Case-5 Lissajous figures

(Particle is subjected to two mutually perpendicular SHMs simultaneously)

We can write the equation

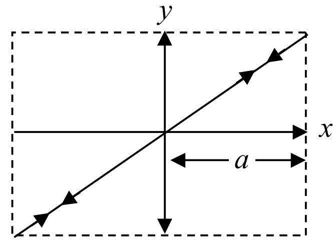
$$x = a\sin(\omega t+\phi),\ \ y = b\sin(\omega t)$$

$$x = a\sin(\omega t)\cos(\phi) + a\cos(\omega t) = a(\sin(\omega t)\cos(\phi)+\cos(\omega t)\sin(\phi))$$

$$\frac{x}{a} = \{\frac{y}{b}\mathrm{os}(\phi) + \{\sqrt{(1-(\frac{y}{b})^2\}}\sin(\phi)\}$$

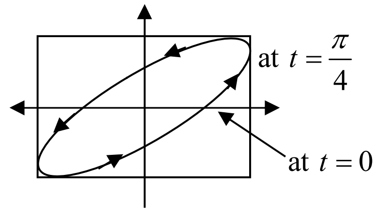
$$[\frac{x}{a}-\frac{y}{b}\mathrm{os}(\phi)]^2 = \{\sqrt{(1-(\frac{y}{b})^2\}}\sin(\phi)\}^2$$

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 - \frac{2xy}{ab}\cos(\phi) = \sin^2\phi\ldots\ldots\ldots..(1)$$



Case-1 $\phi = 0^0,\ y = \frac{b}{a}x \Rightarrow \text{Equation of straight line}$

The trajectory will be straight line



Case-2 $\phi = 45^0,\ \sin\phi = \cos\phi = 1/\sqrt{2}$

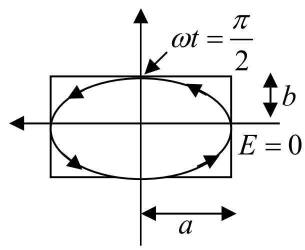
$$x = a\sin\left(\omega t + \frac{\pi}{4}\right)$$

$$y = b\sin(\omega t)$$

At $\omega t = 0,\ x = \frac{a}{\sqrt{2}},\ y = 0$

At $\omega t = \frac{\pi}{4},\ x = a,\ y = \frac{b}{\sqrt{2}}$

The trajectory of motion is oblique ellipsoid in figure below



Case-3 $\phi = \frac{\pi}{2},\ \sin\phi = 1,\ \cos\phi = 0$

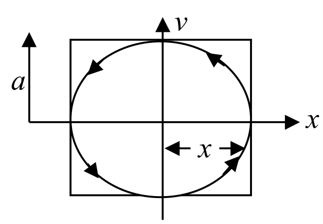
$$x = a\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$y = b\sin(\omega t)$$

At $\omega t = 0,\ x = a,\ y = 0$

At $\omega t = \frac{\pi}{4},\ x = \frac{a}{\sqrt{2}},\ y = \frac{b}{\sqrt{2}}$

The trajectory of motion is shown ellipsoid



Case- 4 When $\phi = \frac{\pi}{2}, and\ \ a = b$

$$\sin\phi = 1,\ \cos\phi = 0$$

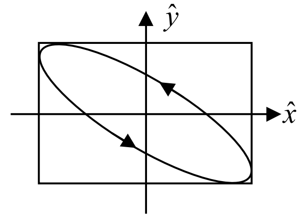
$$x = a\sin\left(\omega t + \frac{\pi}{2}\right),\ y = a\sin(\omega t)$$

At $\omega t = 0,\ x = a,\ y = 0$

At $\omega t = \frac{\pi}{4},\ x = \frac{a}{\sqrt{2}},\ y = \frac{a}{\sqrt{2}}$

The trajectory of motion is shown below

Case-5 When $\phi = \frac{3\pi}{4}$



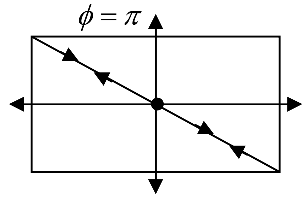
$$x = a\sin\left(\omega t + \frac{3\pi}{4}\right), \quad y = b\sin(\omega t)$$

At $\omega t = 0$, $x = \frac{a}{\sqrt{2}}$, $y = 0$

At $\omega t = \frac{\pi}{4}$, $x = 0$, $y = \frac{b}{\sqrt{2}}$

The trajectory of motion is shown below

$\phi = \pi$

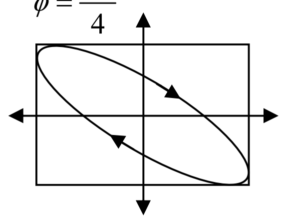


Case-6 $\phi = \pi,\ y = -\frac{b}{a}x \Rightarrow Equation\ of\ \ straight\ line$

The trajectory is shown below.

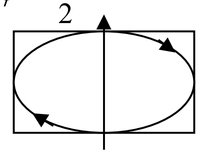
Case-7 $\phi = \frac{5\pi}{4}$

$\phi = \frac{5\pi}{4}$



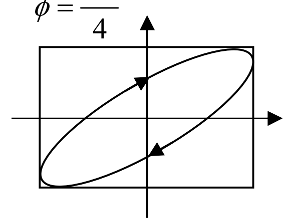
Case 8: $\phi = \frac{3\pi}{2}$ -Direction will be opposite

$\phi = \frac{3\pi}{2}$



Case 9: $\phi = \frac{7\pi}{4}$ -Direction will be opposite

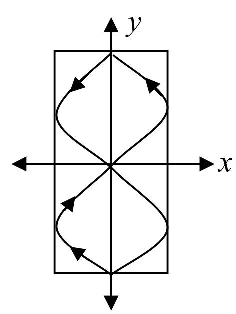
$\phi = \frac{7\pi}{4}$



Now consider the following situation

$x = a\sin(2\omega t + \phi)$

$y = b\sin(\omega t)$



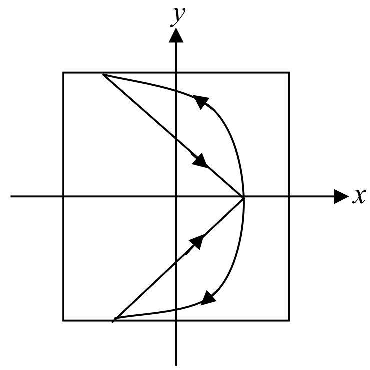
The general solution of these two equations will provides

$$\{\frac{x}{a} - \sin(\phi)\}^2 + 4\frac{y^2}{b^2}(\frac{y^2}{b^2} - 1 + \frac{x}{a}\cos\phi)$$

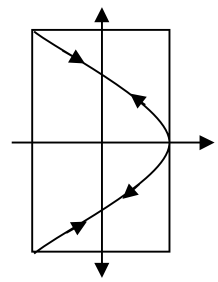
The trajectory of motion is show for different phase

When $\phi = 0$

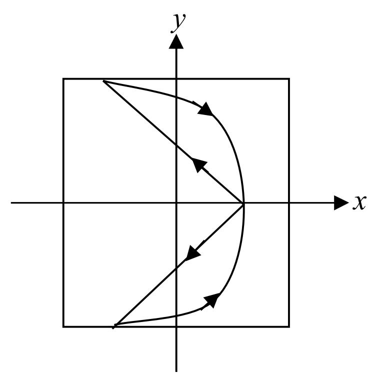
When $\phi = \frac{\pi}{4}$



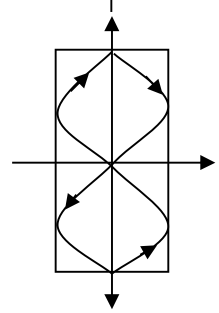
When $\phi = \frac{\pi}{2}$



When $\phi = \frac{3\pi}{4}$



When $\phi = \pi$



When $\phi = \frac{3\pi}{2}$