Jest 2024

Q1. Three equal charges $+q$ are placed at the corners of an equilateral triangle. A test charge constrained to move on the plane of the triangle is placed at the centre of the triangle. Which of the following statements about the test charge is true?

A. Stability of the equilibrium depends on the sign of the test charge.

B. It is in a stable equilibrium.

C. It is not in an equilibrium.

D. It is in an unstable equilibrium.

Topic: Electromagnetic theory

Subtopic: Electrostatics

Ans.: (a)

Solution: For a positive test charge ($+q_{test}$), the equilibrium at the center is unstable.

For a negative test charge ($-q_{test}$), the equilibrium at the center is stable.

Q2. A classical system has the following action:

$$S = \int (\dot{q}^2 + \alpha q\dot{q} + \beta q^2\dot{q})dt$$

where q is the generalized coordinate, and α and β are constants. Which of the following statements is true about the dynamics of the system?

A. The dynamics is independent of α and β

B. The dynamics depends only on α.

C. The dynamics depends only on β

D. The dynamics depends on the ratio $\frac{\alpha}{\beta}$

Topic: Classical mechanics

Subtopic: Lagrangian

Ans.: (a)

Solution: $L = \dot{q}^2 + \alpha q\dot{q} + \beta q^2\dot{q}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \Rightarrow \frac{d}{dt}\left(2\dot{q} + \alpha q + \beta q^2\right) - \alpha\dot{q} - 2\beta q\dot{q} = 0$$

$2\ddot{q} + \alpha\dot{q} + 2\beta q\dot{q} - \alpha\dot{q} - 2\beta q\dot{q} = 0 \Rightarrow \ddot{q} = 0$ so, q is independent of α and β

Q3. A and B are 2×2 Hermitian matrices. $|a_1\rangle$ and $|a_2\rangle$ are two linearly independent eigenvectors of A. Consider the following statements:

1. If $|a_1\rangle$ and $|a_2\rangle$ are eigenvectors of B, then $[A, B] = 0$.

2. If $[A, B] = 0$, then $|a_1\rangle$ and $|a_2\rangle$ are eigenvectors of B.

Mark the correct option.

A. Statement 1 is true but statement 2 is false.

B. Statement 2 is true but statement 1 is false.

C. Both statements 1 and 2 are true.

D. Both statements 1 and 2 are false.

Topic: Quantum mechanics

Subtopic: Commutation

Ans.: (a)

Solution: Assume $A|\alpha_1\rangle = \alpha_1|\alpha_1\rangle, B|\alpha_1\rangle = b_1|\alpha_1\rangle$

If $[A, B] = 0 \Rightarrow AB|\alpha_1\rangle = BA|\alpha_2\rangle \Rightarrow \alpha_1 b_1|\alpha_1\rangle = b_1\alpha_1|\alpha_1\rangle$

Q4. An ideal gas initially at pressure P_i undergoes the following sequence of processes:

1. A reversible adiabatic expansion that doubles its volume.

2. A reversible isothermal compression that restores its original volume.

3. A reversible isothermal expansion that doubles its volume.

4. A reversible adiabatic compression that restores its original volume.

If the final pressure of the gas is P_f, which of the following is true?

A. $P_f = P_i$.

B. $P_f > P_i$.

C. $P_f < P_i$.

D. The relation between P_f and P_i depends on the initial conditions.

Topic: Thermodynamics & Statistical mechanics

Subtopic: First Law of thermodynamics

Ans.: (a)

Solution: Adiabatic expansion (V_i → 2V_i)

For adiabatic: $PV^\gamma = \text{const}$

$$P_a = P_i\left(\frac{V_i}{2V_i}\right)^\gamma = P_i(1/2)^\gamma$$

Isothermal compression ($\mathbf{2\ V\_i \rightarrow V\_i}$)

For isothermal:

$$PV = \text{const}$$

$$P_b = P_a \times \frac{2V_i}{V_i} = 2P_a = 2P_i(1/2)^\gamma$$

Isothermal expansion ($\mathbf{V\_i \rightarrow 2V\_i}$)

Again isothermal:

$$P_c = P_b \times \frac{V_i}{2V_i} = \frac{1}{2}P_b = \frac{1}{2} \times 2P_i(1/2)^\gamma = P_i(1/2)^\gamma$$

Adiabatic compression **($\mathbf{2\ V\_i \rightarrow V\_i}$)**

For adiabatic: $P_f = P_c\left(\frac{2V_i}{V_i}\right)^\gamma = P_i(1/2)^\gamma \times 2^\gamma = P_i$

Q5. A quantum oscillator with energy levels

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega,\ n = 0,1,2\cdots$$

is in equilibrium at a low enough temperature T so that the occupation of all states with $n \geq 2$ is negligible. What is the mean energy of the oscillator as a function of the inverse temperature $\beta = \left(\frac{1}{k_B T}\right)$?

A. $\hbar\omega\left[\frac{1}{2} + \frac{1}{1+\exp(\beta\hbar\omega)}\right]$

B. $\hbar\omega\left[\frac{1}{2} + \frac{1}{1-\exp(\beta\hbar\omega)}\right]$

C. $\hbar\omega[1 + \exp(-\beta\hbar\omega)]$

D. $\hbar\omega[1 - \exp(-\beta\hbar\omega)]$

Topic: Thermodynamics &Statistical mechanics

Subtopic: Canonical Ensemble

Ans.: (a)

Solution: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

$$Z = \sum_n e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega} = e^{-\beta\hbar\omega/2} + e^{-3\beta\hbar\omega/2}$$

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = -\frac{1}{e^{-\beta\hbar\omega/2} + e^{-3\hbar\omega/2}}\left[-\frac{\hbar\omega}{2}e^{-\beta\hbar\omega/2} - \frac{3\hbar\omega}{2}e^{-3\beta\hbar\omega/2}\right]$$

$$= \frac{\hbar\omega}{2}\left(\frac{e^{-\beta\hbar\omega/2} + 3e^{-3\beta\hbar\omega/2}}{e^{-\beta\hbar\omega/2} + e^{-3\beta\hbar\omega/2}}\right) = \frac{\hbar\omega}{2}\left(1 + \frac{2e^{-3\beta\hbar\omega/2}}{e^{-\beta\hbar\omega/2} + e^{-3\beta\hbar\omega/2}}\right)$$

$$= \frac{\hbar\omega}{2}\left(1 + \frac{2}{e^{\beta\hbar\omega} + 1}\right) = \hbar\omega\left(\frac{1}{2} + \frac{1}{1 + e^{\beta\hbar\omega}}\right)$$

Q6. A particle moving in one dimension has the wave function

$$\psi(x) = \exp\left[-\alpha\left(x - \frac{ik_0}{\alpha}\right)^2\right]\sin^2(k_1 x),$$

where α is real positive and k_0, k_1 are real. The expectation value of momentum is

A. $2\hbar k_0$ B. 0 C. $\hbar k_0$ D. $\hbar k_1$

Topic: Quantum Mechanics

Subtopic: Free Particle

Ans.: (a)

Solution: $\psi(x) = \exp\left[-\alpha\left(x^2 - \frac{k_0^2}{\alpha^2} - \left(\frac{2ixk_0}{\alpha}\right)\right)\right]\sin^2 kx \;= \exp -\alpha\left(x^2 - \frac{k_0^2}{a^2}\right)\sin^2 kx \exp 2ik_0 x$

$\langle p\rangle = 2\hbar k_0$

Q7. Two classical particles moving in three dimensions interact via the potential

$$V = K[(x_1^2 + y_1^2) + (x_2^2 + y_2^2) + (z_1 - z_2)^2],$$

where K is a constant, and (x_1, y_1, z_1) and (x_2, y_2, z_2) are the Cartesian coordinates of the two particles. Let (p_1^x, p_1^y, p_1^z) and (p_2^x, p_2^y, p_2^z) be the components of the linear momenta of the two particles, and (L_1^x, L_1^y, L_1^z) and (L_2^x, L_2^y, L_2^z) the components of the corresponding angular momenta. Which of the following statements is true?

A. L_1^z, L_2^z, and $(p_1^z + p_2^z)$ are conserved.

B. L_1^z and L_2^z are not separately conserved but $L_1^z + L_2^z$ is conserved.

C. $(L_1^x + L_2^x), (L_1^y + L_2^y), (L_1^z + L_2^z)$ are conserved.

D. $(L_1^x + L_2^x)$ and $(L_1^y + L_2^y)$ are conserved.

Topic: Classical mechanics

Subtopic: Hamiltonian

Ans.: (a)

Solution: $H = \frac{p_{1x}^2}{2m} + \frac{p_{1y}^2}{2m} + \frac{p_{1z}^2}{2m} + \frac{p_{2x}^2}{2m} + \frac{p_{2y}^2}{2m} + \frac{p_{2z}^2}{2m} + K[(x_1^2 + y_1^2) + (x_2^2 + y_2^2) + (z_1 - z_2)^2]$

The potential is symmetric in $x - y$ plane so angular momentum of z component is conserve so L_1^z and L_2^z is conserve

Now $\frac{\partial H}{\partial z_1} = -\dot{p}_{1z} = 2(z_1 - z_2) \Rightarrow \dot{p}_{1z} = 2(z_2 - z_1)$

$\frac{\partial H}{\partial z_2} = -\dot{p}_{2z} = -2(z_1 - z_2) \Rightarrow \dot{p}_{1z} = -2(z_2 - z_1)$, $\dot{p}_{1z} + \dot{p}_{2z} = 0 \Rightarrow p_{1z} + p_{2z} = c$

Q8. A cylindrical rigid block has principal moments of inertia I about the symmetry axis and $2I$ about each of the perpendicular axes passing through the center of mass. At some instant, the components of angular momentum about the center of mass in the body-fixed principal axis frame is (l, l, l), with $l > 0$. What is the cosine of the angle between the angular momentum and the angular velocity?

A. $\frac{2\sqrt{2}}{3}$ B. $\frac{2}{\sqrt{6}}$ C. $\frac{2}{3}$ D. $\frac{5}{3\sqrt{3}}$

Topic: Classical mechanics

Subtopic- Moment of Inertia

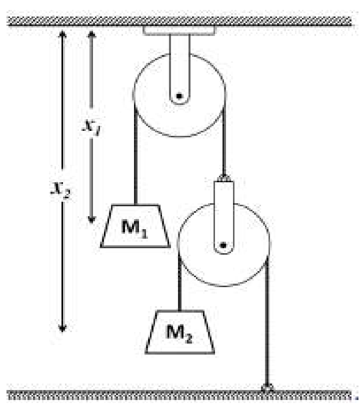
Ans.: (a)

Solution: $I = \begin{pmatrix} 2I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & I \end{pmatrix}, l = I\omega, \Rightarrow l\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$

$$\omega_x = \frac{l}{2I}, \omega_y = \frac{l}{2I} = \omega_z = \frac{l}{I}$$

$$\cos\theta = \frac{\bar{L}\cdot\bar{\omega}}{|\bar{L}||\bar{\omega}|} = \frac{l\frac{l}{2I} + l\cdot\frac{l}{2I} + l\cdot\frac{l}{I}}{\sqrt{3}l.\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{1}}\frac{l}{I}} = \frac{2\sqrt{2}}{3}$$

Q9. Consider a mass-pulley system as shown in the figure. The heights of the blocks as measured from the ceiling are x_1 and x_2, as shown in the figure.



What is the constraint between x_1 and x_2 ?

A. $x_2 + 2x_1 =$ constant

B. $x_2 - x_1 =$ constant

C. $x_2 + x_1 =$ constant

D. They are unconstrained

Topic: Classical mechanics

Subtopic: Equation of Constrain

Ans.: (a)

Solution: $(x_1 - l_1) + l_2 - l_1 = constant$ (i)

$x_2 - l_2 + l_3 - l_2 = constant$ (ii)

$2 \times (i) + (i)$

$2x_1 - 2l_1 + 2l_2 - 2l_1 + x_2 - l_2 + l_3 - l_2 =$ constant

$\Rightarrow 2x_1 + x_2 =$ constant

Q10. Let q and p be the canonical phase space coordinates of a system, where q is the generalized coordinate and p is the generalized momentum. Let us make a transformation of the generalized coordinate as $Q = q^2$. Which of the following functions is canonically conjugate to Q ?

A. $\frac{p}{2q}$ B. $\frac{p}{q}$ C. p^2 D. $\frac{p^2}{2q^2}$

Topic: Classical Mechanics

Subtopic: Poisson Bracket

Ans.: (a)

Solution: $\{Q, P\} = 1$

$\Rightarrow \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial p}{\partial q} = 1 \qquad 2q.\frac{\partial P}{\partial p} - 0 = 1 \Rightarrow P = \frac{p}{2q}$

Q11. A solid sphere of radius R has a volume charge density $\rho = \rho_0 \sin\ 2\theta$. How does the leading term in the electrostatic potential depend on the distance r far away from the charged sphere?

A. $\frac{1}{r^2}$ B. $\frac{1}{r}$ C. r D. Does not depend on r

Topic: Electromagnetic Theory

Subtopic: Electrostatics

Ans. : (a)

Solution: We write this as $V(\mathbf{r}) = V_{\text{mon}}(\mathbf{r}) + V_{\text{dip}}(\mathbf{r}) + V_{\text{quad}}(\mathbf{r}) + \cdots$, where,

$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r}\int_{V'} \rho(\mathbf{r}')dV'$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2}\int_{V'} \rho(\mathbf{r}')(\hat{\mathbf{r}} \cdot \mathbf{r}')dV'$$

$$V_{\text{mon}} = \frac{1}{4\pi\epsilon_0 r}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho_0 \sin 2\theta\, r^2 dr \sin\theta d\theta d\phi = 0$$

$$V_{\text{dip}} = \frac{1}{4\pi\varepsilon_0 r^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho_0 \sin 2\theta\, r^2 \sin\ \theta \cos\ \theta d\theta d\phi dr \neq 0$$

$V_{\text{dip}}(\mathbf{r}) = \frac{1}{r^2}$

Q12. Let $(G,\circ)$ be a discrete group of order 4 where the group operation ' $\circ$ ' among the various elements of $G=\{e,a,b,c\}$ is given by the following multiplication table:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Which of the following is correct?

A. $(G,\circ)$ is non-cyclic and abelian

B. $(G,\circ)$ is cyclic and abelian

C. $(G,\circ)$ is cyclic and non-abelian

D. $(G,\circ)$ is non-cyclic and non-abelian

Topic: Mathematical Physics

Subtopic: Group Theory

Ans.: (a)

Solution: $a \cdot b = c$

$b \cdot a = c$

$a \cdot b = b \cdot a$ Abelian

nm-cyclic there is not a single element which can generate all the moments of the group.

Q13. Consider the Fourier transform of a function $f(x)$ defined as

$$g(p)=\int_{-\infty}^{\infty} f(x)\exp(ipx)dx, \text{ where } f(x)=\frac{1}{\sqrt{|x|}}$$

Which of the following is the correct form of $g(p)$ for some constant β ?

A. $g(p)=\frac{\beta}{\sqrt{|p|}}$

B. $g(p)=\frac{\beta}{p}$

C. $g(p)=\frac{\beta}{p^2}$

D. $g(p)=\frac{\beta}{|p|}$

Topic: Mathematical Physics

Subtopic: Fourier Transformation

Ans.: (a)

Solution: $g(p)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{|x|}}\cdot e^{ipx}dx=2\int_0^{\infty}\frac{1}{\sqrt{x}}\cdot\cos pxdx$

$px=t^2, \qquad pdx=2tdt$

$=2\int_0^{\infty}\frac{\sqrt{p}}{t}\cos t^2\frac{2tdt}{p}pdx=4\frac{1}{\sqrt{p}}\int_0^{\infty}\cos t^2dt$ (Fresnel's Integral)

$$= 4\frac{1}{\sqrt{p}}\sqrt{\frac{\pi}{8}} = \sqrt{\frac{2\pi}{p}} = \frac{\beta}{\sqrt{|p|}}$$

Q14. What is the power of a light source emitting photons of wavelength 600 nm at the rate of one photon per second? Planck's constant $h = 6.6 \times 10^{-34}$ Joule sec and speed of light $c = 3 \times 10^{8}\ \mathrm{m/sec}$.

A. 3.3×10^{-19} W B. 3.3×10^{-18} W C. 6.0×10^{-19} W D. 6.0×10^{-18} W

Topic-Modern Physics

Subtopic-Particle Nature of Light

Ans.: (a)

Solution: $E = hv = hc/\lambda = \frac{6.6\times10^{-34}\times3\times10^{8}}{610\times10^{9}}$ J

$$P = \frac{E}{t} = \frac{6.6 \times 10^{-34} \times 3 \times 10^{8}}{6\mathrm{w} \times 10^{-9} \times 1\ \mathrm{s}} = 3.3 \times 10^{-19}\mathrm{w}$$

Q15. What is the right sequence of optical components to convert unpolarized light into circularly polarized light?

A. Light source → linear polarizer → quarter wave plate

B. Light source → quarter wave plate → half wave plate

C. Light source → linear polarizer → half wave plate

D. Light source → half wave plate → quarter wave plate

Topic: Optics

Subtopic: Polarization

Ans.: (a)

Solution: Light more → Linear (unpolarized polarizer light) (preanizes) ↓ quarter wave plate (used to convert-

linearly polarised light to circularly preanzed light)

Q16. Where does the Fermi level of an n-type semiconductor lie?

A. Near the conduction band minimum B. Near the valence band maximum

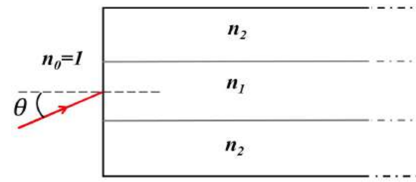
C. At the middle of the energy gap D. Inside the valence band

Topic: Electronics

Subtopic: Semiconductor

Ans.: (a)

Q17. A step index optical fiber has refractive indices $n_1 = 1.474$ for core region and $n_2 = 1.470$ for the cladding region. A ray of light is incident from air into the core through the cross section of the fiber at an angle θ with the normal. What is the limiting value of θ below which the light ray will be totally internally reflected? Refractive index of air is taken as 1.



A. $6.229°$ B. $58.194°$ C. $2.862°$ D. $4.222°$

Topic: Optics

Subtopic: Fiber Optics

Ans.: (a)

Solution: $NA =$ Numerical Aperture $= \sqrt{n_1^2 - n_2^2} = 0.108517279.$

$$\sin\ \theta = NA$$

$$\Rightarrow \theta = \sin(NA) = 6.229°$$

Q18. A satellite of mass 2000 kg is placed in an elliptic orbit around Earth with semi major axis A. Assume that the total energy of the orbiting satellite is E and the angular momentum is L. Through a series of manoeuvres, the elliptic orbit is changed to a circular orbit with radius A. For the orbit change described, which of the following is true?

A. E does not change, but L changes B. E changes, but L does not change

C. Both E and L change D. Neither E nor L changes

Topic-Classical Mechanics

Subtopic: Central Force Problem

Ans.: (a)

Solution: $E_1 = \frac{-k}{2A} E_2 = -\frac{k}{2A}$ (after elliptic orbit is changed to circular orbit)

No change in energy.

So angular momentum must change

Q19. Which of the following functions is not a valid thermodynamic function of internal energy U in terms of entropy S, volume V, and number of particles N ? Here U_0, α, β, A, B and C are constants.

A. $U_0 \exp\left(\frac{\alpha V^2 N}{S^2}\right)$ B. $\left(\frac{AV^2}{N}\right) \exp\left(\frac{\beta VN}{S^2}\right)$ C. $\frac{BS^2V^2}{N^3}$ D. $\frac{CN^2}{\sqrt{SV}}$

Topic: Thermodynamics and statistical mechanics

Subtopic: Thermodynamical potential

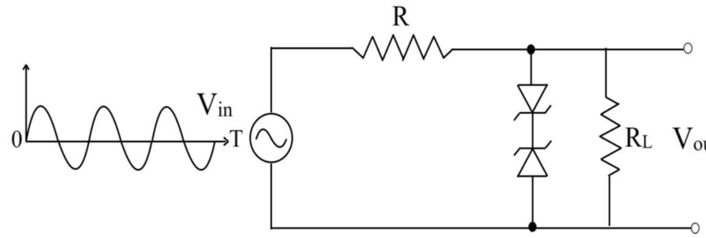
Ans.: (a)

Solution: $V \to XV, N \to XN, S \to XS$

$$u_0 \exp\left(\frac{\alpha x^2 v^2 x N}{x^2 s^2}\right), = u_0 \exp\left(\frac{\alpha \lambda v^2 N}{s^2}\right), = u_0 \exp\left(\frac{\alpha \lambda \ ^2 N}{s^2}\right), = {u_0}^{1-\lambda} U^{\lambda}$$

So, it is nonlinear and can't be a valid function of internal energy.

Q20. What is the output waveform of the circuit for the given input signal? Assume that the Zener diodes are identical, amplitude of the input voltage V_{in} is twice the Zener breakdown voltage, and $R_L = 10R$.



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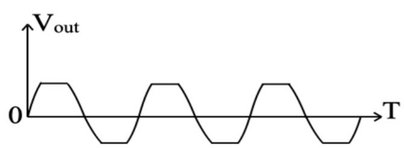

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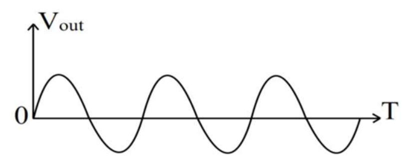

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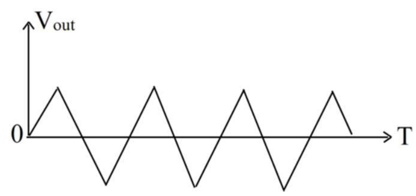


(d)


Topic: Electronics

Subtopic: Zener Diode

Ans.: (a)

Solution: $2V_z = V_{in}$

$V_z = 11 \text{ V}$

$-V_{in} + IR - V_z + V_z = 0$

Q21. Given that the latent heat of liquefaction is 80Cal/g, what is the change in entropy when 10 g of ice at 0°C is converted into water at the same temperature?

A. 2.93CalK^{-1} B. 3.42CalK^{-1} C. 2.0CalK^{-1} D. 4.5CalK^{-1}

Topic: Thermodynamics and Statistical Mechanics

Subtopic-Second Law

Ans.: (a)

Solution: $\Delta S = \frac{\Delta Q}{I} = \frac{mL}{T} = \frac{10 \times 80}{273} = 2.93 Cal/K.$

Q22. Consider a system of N noninteracting spin- $\frac{1}{2}$ atoms subjected to a magnetic field with the Hamiltonian given by

$$H=-g\mu_B B\sum_{i=1}^{N} S_i^z,$$

where g is the dimensionless Lande factor, μ_B is the Bohr magneton, B is the strength of the magentic field, and S_i^z is the z-component of the spin of the i th atom (S_i^z takes values $\pm\frac{1}{2}$). The system is in equilibrium at temperature T. What is the probability that the z-component of the spins corresponding to two given atoms have the same value? Take $\beta=\frac{1}{k_B T}$, where k_B is the Boltzmann constant.

A. $\frac{\exp(-\beta g\mu_B B)+\exp(\beta g\mu_B B)}{2+\exp(-\beta g\mu_B B)+\exp(\beta g\mu_B B)}$

B. $\frac{\exp(-\beta g\mu_B B)}{2+\exp(-\beta g\mu_B B)+\exp(\beta g\mu_B B)}$

C. $\frac{\exp(\beta g\mu_B B)}{2+\exp(-\beta g\mu_B B)+\text{ex}\quad(\beta g\mu_B B)}$

D. $\frac{1}{4}$

Topic: Thermodynamics and Statistical Mechanics

Subtopic: Canonical Ensemble

Ans.: (a)

Solution: $z=e^{-\beta g\mu_B B\left(-\frac{1}{2}-\frac{1}{2}\right)}+e^{-\beta g\mu_B B\left(\frac{1}{2}+\frac{1}{2}\right)}+e^{-\beta g\mu_B B\left(-\frac{1}{2}+\frac{1}{2}\right)}+e^{-\beta g\mu_B B\left(\frac{1}{2}-\frac{1}{2}\right)}$

$=2+e^{+\beta g\mu_B B}+e^{-\beta g\mu_B B}$

Prob $=\frac{e^{-\beta g\mu_B B}+e^{\beta g\mu_B B}}{2+e^{\beta g\mu_B B}+e^{-\beta g\mu_B B}}$

Q23. Two electrons have orbital angular momentum quantum numbers $l_1=3$ and $l_2=2$, respectively. Let $L^z=L_1^z+L_2^z$, where L_1^z and L_2^z are the z-components of the respective angular momentum operators. How many linearly independent states have L^z quantum number $m=2$?

A. 4   B. 3   C. 11   D. 0

Topic: Quantum mechanics

Subtopic: Angular momentum

Ans.: (a)

Solution: $l_1=3\ m_{l_1}=-3,-2,-1,0,1,2,3$

$l_2=2\ m_{l_2}=-2,-1,0,1,2$

For, $m_l=m_{l_1}+m_{l_2}\quad m_{l_1}+m_{l_2}=2\quad \left(m_{l_1},m_{l_2}\right)=(0,2),(1,1),(2,0),(3,-1)$

Q24. A quantum particle is subjected to the potential $V(x)=ax+bx^2$, where a and b are constants. What is the mean position of the particle in the first excited state?

A. $-\frac{a}{2b}$ B. $\frac{a}{2b}$ C. $-\frac{a}{b}$ D. $\frac{a}{b}$

Topic: Classical Mechanics

Subtopic: Stability Analysis

Ans.: (a)

Solution: $V=bx^2+ax=b\left(x^2+\frac{2ax}{2b}+\frac{a^2}{4b^2}-\frac{a^2}{4b^2}\right)=b\left(x+\frac{a}{2b}\right)^2-\frac{a^2}{4b}$

So potential is centred at $x=-\frac{b}{2a}$ so average value of $\langle x\rangle=-\frac{b}{2a}$

Q25. The density of states of a system of N particles at energy E is

$$g(E,N)=\begin{cases}\frac{1}{(\hbar\omega)^N}\frac{E^{N-1}}{(N-1)!} & \text{for } E\geq 0\\ 0 & \text{for } E<0\end{cases}$$

where $\hbar$ is the Planck's constant and ω is a natural frequency of the system. Taking k_B to be the Boltzmann constant, compute the temperature of the system at energy E.

A. $\frac{E}{Nk_B}$ B. $\frac{1}{k_B}\left(\frac{E}{N}+\frac{1}{2}\hbar\omega\right)$ C. $\frac{1}{k_B}\left(\frac{E}{N}+\hbar\omega\right)$ D. $\frac{1}{k_B}\sqrt{\left(\frac{E}{N}\right)^2+(\hbar\omega)^2}$

Topic: Thermodynamics and Statistical Mechanics

Subtopic- Microcanonical

Ans.: (a)

Solution: $S=k_B\ln\,\Omega.$

$\Rightarrow S=k_B\ln\frac{1}{(\hbar\omega)^N}\frac{E^{N-1}}{(N-1)!}$

$\Rightarrow TdS=dE$

$\Rightarrow\frac{1}{T}=\frac{dS}{dE}=\frac{K_B(\hbar\omega)^N N-1)!}{E^{N-1}}\frac{1}{(\hbar\omega)^N}(N-1)\frac{E^{N-2}}{(N-1)!}$

$\therefore\frac{K_B}{E}(N-1)=\frac{NK_B}{E}$, If N is very Large $N-1=N$, $T=\frac{E}{NK_B}$

Part B: 3-Mark MCQ

Q1. The energy spectrum for a system of spinless noninteracting fermions consists of $(N+1)$ nondegenerate energy levels $0, \varepsilon, 2\varepsilon, \ldots, N\varepsilon (\varepsilon > 0)$. Let $x = \exp\left(-\frac{\varepsilon}{k_B T}\right)$, where k_B is the Boltzmann constant and T is the temperature. For N identical fermions in thermal equilibrium at temperature T, what is the average occupancy of the highest energy level?

A. $\frac{x - x^{N+1}}{1 - x^{N+1}}$ B. $\frac{x - x^{N+1}}{1 + x^{N+1}}$ C. $\frac{x}{1 - x^N}$ D. $\frac{x^N}{1 + x^N}$

Topic: Thermodynamics and Statistical Mechanics

Subtopic: Canonical

Ans.: (a)

Solution: $z = e^{\beta(0)} + e^{-\beta} + e^{-2\beta\varepsilon} + \cdots + e^{-N\beta}$

$$= \frac{1 - e^{-\beta(N+1)\varepsilon}}{1 - e^{-\beta}} = \frac{1 - x^{N+1}}{1 - \mathrm{x}}$$

$$p = \frac{e^{-\beta N\varepsilon}}{z} = \frac{x^{+N}(1-x)}{1 - x^{N+1}} = \frac{x^{\mathrm{N}} - x^{\mathrm{N+1}}}{1 - x^{N+1}}$$

Q2. A two-level quantum system has the Hamiltonian

$$H = \hbar\omega_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At $t = 0$, the system is in the state

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

What is the earliest time $t > 0$ at which a measurement of $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ will yield the value -1 with probability one?

A. $\frac{\pi}{2\omega_0}$ B. $\frac{2\pi}{\omega_0}$ C. $\frac{\pi}{\omega_0}$ D. Never

Topic- Quantum Mechanics

Subtopic: Postulates of Quantum Mechanics

Ans.: (a)

Solution: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle)$

The energy eigen value is $\hbar\omega$ with eigen vector $|\phi_1\rangle = \frac{1}{\sqrt{2}}\binom{1}{1}$ and

The energy eigen value is $-\hbar\omega$ with eigen vector $|\phi_2\rangle = \frac{1}{\sqrt{2}}\binom{1}{-1}$

$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle e^{-i\omega_0 t} + \frac{1}{\sqrt{2}}|\phi_2\rangle e^{i\omega_0 t}$

Measurement of $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is 1 with state $|\phi_{z+}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

And -1 with eigen state is $|\phi_{z-}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

probability to get -1 is $\frac{|\langle \phi_{z-} | \psi(t)\rangle|^2}{\langle \psi | \psi \rangle} = 1 \Rightarrow \frac{\left| \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t} + \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_0 t} \right|^2}{1}$

$\sin^2 \omega_0 t = 1 \Rightarrow \Rightarrow \omega_0 t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega_0}$

Q3. The singular matrix

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix}$$

commutes with the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The eigenvalues of A are

A. $(0,3,11)$ B. $(0,3,13)$ C. $(0,0,12)$ D. $(0,2,5)$

Topic: Mathematical Physics

Sub Topic: Matrices

Ans.: (a)

Solution: Given singular matrix

$\text{def}[A] = 0$

$\Rightarrow$ Product of eigenvalues $= 0$

Sum of eigenvalues $= 14$

Q4. A polynomial $C_n(x)$ of degree n defined on the domain $x \in [-1,1]$ satisfies the differential equation

$$(1-x^2)\frac{d^2C_n}{dx^2} - x\frac{dC_n}{dx} + n^2C_n = 0.$$

The polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} \sigma(x)C_n(x)C_m(x)dx = 0$$

for $n \neq m$. What is $\sigma(x)$?

A. $(1-x^2)^{-1/2}$ B. $(1-x^2)$ C. 1 D. $\exp(-x^2)$

Topic: Mathematical physics

Sub Topic: Special Function

Ans.: (a)

Solution: $\sigma(x) = \frac{1}{1-x^2}e^{\int \frac{-x}{1-x^2}dx} = \frac{1}{1-x^2}e^{\int \frac{dt}{2t}}$

$$= \frac{1}{1-x^2}e^{\ln\sqrt{1-x^2}} = \frac{1}{1-x^2}\sqrt{1-x^2} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

Q5. Consider a particle of mass m and nonzero angular momentum ℓ subjected to a central force potential $V(r) = k\ln r$, where k is a positive constant. What is the radius R at which it can have a circular orbit? Will the circular orbit be stable or unstable?

A. $R = \frac{\ell}{\sqrt{km}}$ and stable B. $R = \frac{\ell}{\sqrt{km}}$ and unstable

C. $R = \frac{\ell}{\sqrt{2km}}$ and stable D. $R = \frac{\ell}{\sqrt{2km}}$ and unstable

Topic: Classical mechanics

Sub Topic: Central Force Problem

Ans.: (a)

Solution: $V_{effect} = \frac{L^2}{2mr^2} + k\ln r$

$$\frac{dV_{effect}}{dr} = 0 = -\frac{L^2}{mr^3} + \frac{k}{r} = 0 \Rightarrow r = \left(\frac{L^2}{mk}\right)^{1/2}$$

$$\frac{dV_{effect}}{dr^2} = \frac{3L^2}{mr^4} - \frac{k}{r^2} \text{ at } r = \left(\frac{L^2}{mk}\right)^{1/2}, \frac{d^2V_{eff}}{dr^2} = \frac{2mk}{L^2} > 0$$

So, stable

Q6. Consider a particle of mass m moving in a three-dimensional delta-function potential well $V(\vec{r}) = -\alpha\delta^3(\vec{r})$, where $\alpha > 0$. Which of the following is an allowed expression for the energy of a bound state for some dimensionless proportionality constant $\beta > 0$?

A. $-\frac{\beta\hbar^6}{\alpha^2 m^3}$ B. $\frac{\beta\hbar^6}{\alpha^2 m^3}$ C. $-\frac{\beta\alpha^2 m}{\hbar^2}$ D. $\frac{\beta\alpha^2 m}{\hbar^2}$

Topic: Mathematical Physics

Sub Topic: Dirac Delta

Ans.: (a)

Solution: $\dim\alpha = \dim\frac{V}{\delta^3(\vec{r}),} = [ML^2T^{-2}]L^3 = [ML^5T^{-2}]$

$$\frac{-\beta h^6}{\alpha^2 m^3} = \dim\left(\frac{h^6}{\alpha^2 m^3}\right) = \frac{[ML^2T^{-1}]^6}{[ML^5T^{-2}]^2 M^3} = \frac{[M^6L^{12}T^{-6}]}{[M^5L^{10}T^{-4}]}$$

$= [ML^2 \cdot T^{-2}]$ =dim of energy.

Q7. A particle with energy $E > 0$ is incident from the right $(x > 0)$ on a one-dimensional potential composed of a delta-function barrier at $x = 0$ and a hard wall at $x = -a$:

$$V(x) = \begin{cases} \alpha\delta(x), & x > -a \\ \infty, & x \le -a \end{cases}$$

where $\alpha > 0$ and $a > 0$. Let us define $\kappa^2 = \frac{2m}{\hbar^2}$ and the dimensionless quantities: $\xi = \kappa a$ and $\beta = \frac{\hbar^2}{2m\alpha a}$. For some energy E the particle reflects from the barrier without any phase shift. Which of the following transcendental equations determines this energy? [Note that in the presence of the delta function barrier, the derivative of the wave function has a discontinuity at $x = 0$: $\psi'(0^+) - \psi'(0^-) = \frac{\psi(0)}{\beta a}$.]

A. $\tan\xi = -\beta\xi$ B. $\tan\xi = \beta\xi$ C. $\tanh\xi = \beta\xi$ D. $\tanh\xi = -\beta\xi$

Topic: Quantum Mechanics

Sub Topic: Dirac Delta

Ans.: (a)

Solution: $\psi_I = Ae^{ikx} = A\cos kx$

$\psi_{II} = C\sin kx + D\cos kx$, $\psi_{III} = 0$, $\psi_I|_{x=0} = \psi_{II}|_{x=0} \Rightarrow A = D$

There is a hand wall at $x = -a$.

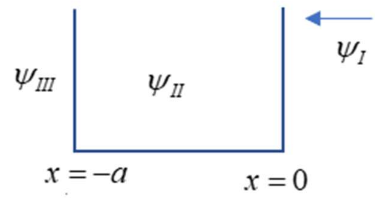
$\psi_{\text{II}}(x = -a) = 0$

$$\Rightarrow (\sin k(-a) + D\cos(+k(-a) = 0 \Rightarrow D\cos ka - c\sin ka = 0 \Rightarrow \frac{D}{c} = \tan ka$$

$\frac{-h^2}{2m}\int_{0-\epsilon}^{0+\epsilon}\frac{d^2\psi}{dx^2}dx+\int_{0-\epsilon}^{0+\epsilon}\delta(x)+(x)dx=\int_{0-\epsilon}^{0+\epsilon}$

$\Rightarrow -\frac{\hbar^2}{2m}\psi[\psi'(0^+)-\psi'(0^-)]=-\alpha\psi(0)$

$\Rightarrow -AK\sin K(0)-CK\cos K(0)+D\sin K(0)=\frac{\tan\alpha}{\hbar^2}H(0)$



$\Rightarrow -CK=\frac{\psi(0)}{\beta a}\Rightarrow -CK=\frac{A}{\beta a}\Rightarrow -CK=\frac{D}{\beta a}\Rightarrow\frac{D}{C}=-\beta Ka$

$\Rightarrow \tan Ka=-\beta Ka\Rightarrow\tan\xi=-\beta\xi$

Q8. The speed distribution of the molecules of an ideal gas in equilibrium at inverse temperature $\beta\left(=\frac{1}{k_BT}\right)$ is found to obey the Maxwell distribution:

$$P(v)=Cv^2\exp\left(-\frac{1}{2}\beta mv^2\right)$$

where m is the mass of a molecule and C is a normalization constant. Compute $(\langle v^4\rangle)^{1/4}$.

A. $\sqrt{\frac{\sqrt{15}k_BT}{m}}$ B. $\sqrt{\frac{4k_BT}{m}}$ C. $\sqrt{\frac{3k_BT}{m}}$ D. $\sqrt{\frac{11\ _BT}{\pi m}}$

Topic: Thermodynamics and kinetic theory of gases

Sub Topic: Kinetic theory of gases

Ans.: (a)

Solution: $C\int_0^\infty v^2e^{-\frac{\beta mv^2}{2}}dv=1$

$\Rightarrow C=\sqrt{\frac{2}{\pi}\left(\frac{m}{k_BT}\right)^{3/2}},\ \langle v^4\rangle=\int_0^\infty\sqrt{\frac{2}{\pi}}\left(\frac{m}{k_BT}\right)^{3/2}v^6e^{-\frac{\beta mv^2}{2}}dv=\sqrt{\frac{2}{\pi}\left(\frac{m}{k_BT}\right)^{3/2}\frac{1}{2}\cdot\frac{\Gamma(7/2)}{\left(\frac{\beta m}{2}\right)^{7/2}}}$

$=15\left(\frac{k_BT}{m}\right)^2,(\langle v^4\rangle)^{1/4}=\sqrt{\sqrt{15}\frac{k_BT}{m}}$

Q9. A classical particle undergoing simple harmonic motion is confined to the region $(-a,a)$ on the X-axis. If a snapshot of the particle is taken at a random instant of time, what is the probability that it would be found in the region $\left(\frac{a}{2},a\right)$?

A. $\frac{1}{3}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{2}{5}$

Topic: Quantum Mechanics

Subtopic: Probability Density

Ans.: (a)

Solution: $E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2 \Rightarrow \frac{1}{2}m\omega^2a^2=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2$

$$\frac{dx}{dt}=\omega\sqrt{a^2-x^2}=\frac{2\pi}{T}\sqrt{a^2-x^2} \Rightarrow \frac{dt}{dT}=p(x\to x+dx)\propto \frac{dx}{\sqrt{a^2-x^2}}$$

$$p(a/2\to a)=\frac{\int_{a/2}^{a}\frac{dx}{\sqrt{a^2-x^2}}}{\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}} \text{ put } x=\sin\theta \text{ then } \frac{\frac{\pi}{2}-\frac{\pi}{6}}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}=\frac{3-1}{6}=\frac{1}{3}$$

Q10. The ratio of the molar specific heats of an ideal gas is $\gamma=\frac{c_p}{c_v}=\frac{3}{2}$. It undergoes a reversible isothermal expansion in which its volume doubles. Next, it undergoes a reversible isochoric process such that the change in entropy of the second process is equal to the change in entropy of the first process. What is the ratio of the final temperature to the initial temperature?

A. $\sqrt{2}$ B. 2 C. 3 D. $\frac{3}{2}$

Topic: Thermodynamics and Statistical mechanics

Subtopic: Second Law of Thermodynamics

Ans.: (a)

Solution: $S_{1-2}=nR\ln 2 \quad S_{2-3}=\frac{nR}{\gamma-1}\ln\frac{T_f}{T_i}$

$$S_{1-2}=S_{2-3}\Rightarrow \ln(2)^{\gamma-1}=\ln\left(\frac{T_f}{T_i}\right)\Rightarrow \frac{T_f}{T_i}=(2)^{\gamma-1}=(2)^{\frac{3}{2}-1}=2^{1/2}$$

Q11. Two trains, each having proper length L_0 are moving towards each other with the same speed v but in opposite directions as measured by an observer in an inertial frame. What is the length of one of the trains as measured by an observer in the other train?

A. $L_0\left(\frac{c^2-v^2}{c^2+v^2}\right)$ B. $L_0\sqrt{\left(\frac{c^2-v^2}{c^2+v^2}\right)}$ C. $L_0\sqrt{1-\frac{v^2}{4c^2}}$ D. $L_0\sqrt{1-\frac{v^2}{c^2}}$

Topic: Classical Mechanics

Subtopic: Special Theory of Relativity

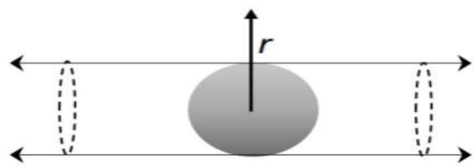
Ans.: (a)

Solution: $v_{12}=\frac{v-(-v)}{1+v^2/c^2}=\frac{2v}{1+v^2/c^2}$

$$L = L_0\sqrt{1-\frac{v_{12}^2}{c^2}} = L_0\sqrt{1-\frac{\left(\frac{2v}{1+v^2/2}\right)^2}{c^2}}$$

$$= L_0\sqrt{1-\frac{4v^2c^2}{(1+v^2c^2)^2}} = L_0\sqrt{1-\frac{4\alpha}{(1+\alpha)^2}}\ \alpha = \frac{v^2}{c^2} = L_0\ \frac{c^2-v^2}{c^2+v^2} =$$

Q12. An infinitely long cylinder of radius R has uniform volume charge density. A spherical region of radius R is carved out of it, as shown in the figure. At what value of r (the radial coordinate in a cylindrical system, with origin at the center of the sphere) is the electric field maximum?



A. $r = \frac{4}{3}R$ B. $r = R$ C. $r = \frac{2}{3}R$ D. $r = \frac{3}{2}R$

Due to a possible alternate interpretation of the question, both Option A and Option B will be treated as correct answer.

Topic: Electromagnetic theory

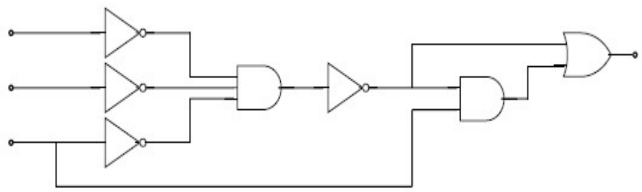
Subtopic: Electrostatics

Ans.: (a)

Solution: $E_{\text{cylinder}} = \frac{\rho R^2}{2\epsilon_0 r}\ r > R, E_{sphere} = \frac{\rho R^3}{3\epsilon_0 r^2}\ r > R,$

$$E_{\text{net}} = \frac{\rho R^2}{2\epsilon_0 r} - \frac{\rho R^3}{3\epsilon_0 r^2}, \frac{dE_{\text{net}}}{dr} = 0,\ r = \frac{4}{3}R$$

Q13. What is the output of the following logic circuit?



A. $X = A$ AND B AND C

B. $X = (A$ OR $C)$AND $(B$ OR $C)$

C. $X = (A$ OR $C)$AND $(B$ OR $C)$ AND C

D. $X = (\bar{A}$ OR $\bar{B}$ OR $\bar{C})$ANDC

This question is withdrawn since the labels are not shown in the figure.

ALL CANDIDATES WILL BE AWARDED 3 MARKS.

Topic: Electronics

Subtopic: Digital Electronics

Ans.: (a)

Solution: $(A+B+C)\cdot(=A+BA+CA=A)$

$CA+B+C+C$

$=AC+BC+C$

$=AC+C(B+1)\ =AC+C=A$

Q14. What is the value of $\int_0^\infty \frac{dx}{1+x^3}$?

A. $\frac{2\pi}{3\sqrt{3}}$ B. $\frac{\pi}{3\sqrt{3}}$ C. $\frac{2\pi}{\sqrt{3}}$ D. $\frac{\pi}{3}$

Topic: Mathematical Physics

Subtopic: Complex Variable

Ans.: (a)

Solution: $\int_0^\infty \frac{dx}{1+x^n}=\frac{\pi/n}{\sin \pi/n}$

$$\int_0^\infty \frac{dx}{1+x^3}=\frac{\pi/3}{\sin\ \pi/3}=\frac{2\pi}{3\sqrt{3}}$$

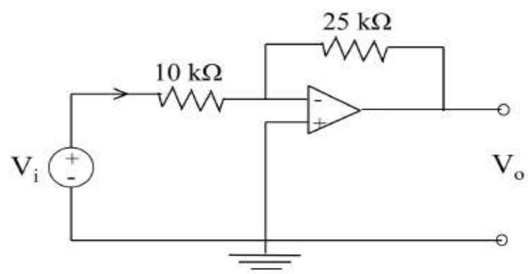
Q15. What is the output voltage V_o and current I in the $10\text{k}\Omega$ resistance of the following circuit? $V_i=0.5$ V.

A. $V_o=-1.25\mathbf{V},\ I=50\mu\mathbf{A}$

B. $V_o=-0.4$ V, $I=50\mu$ A

C. $V_o=-0.4$ V, $I=20\mu$ A

D. $V_o=-1.25$ V, $I=20\mu$ A



Topic: Electronics

Subtopic: Operational Amplifier

Ans.: (a)

Solution: $V_0=-\frac{R_f}{R_i}V_{in}=-\frac{25\times10^3}{10\times10^3}\times0.5=-1.25$ V

$I=\frac{V}{R}=\frac{0.5}{10\times10^3}=50\mu$ A

Part C: 3-Mark Numerical

Q1. A system of two noninteracting identical bosons is in thermal equilibrium at temperature T. The particles can be in one of three states with nondegenerate energy eigenvalues $-\varepsilon, 0$ and ε. The temperature T is such that $\exp\left(-\frac{\varepsilon}{k_B T}\right) = \frac{1}{2}$, where k_B is the Boltzmann's constant. The average energy of the system is found to be $\langle E \rangle = -\frac{n}{35}\varepsilon$, where n is an integer. What is the value of n ?

Topic: Thermodynamics and Statistical Mechanics

Subtopic: Canonical

Ans. : 36 or 15

Solution: $z = e^{+\beta\varepsilon} + 2e^{\beta 0} + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + e^{2\beta\varepsilon}$

$\langle E \rangle = -\frac{1}{z}\frac{\partial z}{\partial \beta} = -\frac{36}{35}\varepsilon, n$ =36

Q2. A semi-infinite, thin wire extending from $-\infty$ to zero along the z-axis carries a constant current I in the positive z-direction. The wire is charge-neutral except at $z = 0$, where the inflowing charge is accumulated. What is the absolute value of the line integral $\frac{4}{\mu_0 I}\oint \vec{B} \cdot d\vec{l}$ along the circle $x^2 + y^2 = 1$? $\vec{B}$ is the magnetic field and μ_0 is the permeability in free space. Assume that the accumulated charge at $z = 0$ is a point charge.

Topic: Electromagnetic Theory

Subtopic: Magnetostatics

Ans. : 2

Solution: $B = \frac{\mu_0 I}{4\pi R}(\sin\phi_1 + \sin\phi_2) = \frac{\mu_0 I}{4\pi R}\sin 90^\circ$

$$\frac{4}{\mu_0 I}\oint \vec{B} \cdot d\vec{l} = \frac{4}{\mu_0 I} \cdot \frac{\mu_0 I}{4\pi R} \cdot 2\pi(R) = 2$$

Q3. Consider the rotation matrix

$$R = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}.$$

Let ϕ be the angle of rotation. What is the value of $\sec^2\phi$?

Topic Mathematical physics

Sub topic: Matrices

Ans.: 4

Solution: $R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\text{Tr}[R(\phi)] = 1 + 2\cos\ \phi, \quad \text{Tr}[\ R] = 2, 1 + 2\cos\ \phi = 2$

$\cos\ \phi = \frac{1}{2}, \quad \sec\phi = 2, \sec^2\ \phi = 4$

Q4. A quantum harmonic oscillator of mass m and angular frequency ω is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|287\rangle + |288\rangle)$, where $|n\rangle$ denotes the n^{th} normalized energy eigenstate of the harmonic oscillator. Let $L_0 = \sqrt{\frac{\hbar}{m\omega}}$ denote the oscillator size and $\langle\hat{x}\rangle$ denote the expectation value of the position operator in the state $|\psi\rangle$. What is the value of $\frac{\langle\hat{x}\rangle}{L_0}$? You may use the form of the position operator in terms of the raising and lowering operators: $\hat{x} = \frac{L_0}{\sqrt{2}}(a + a^\dagger)$.

Topic: Quantum Mechanics

Subtopic: Harmonic Oscillator

Answer: 12

Solution: $\hat{x} = \frac{L_0}{\sqrt{2}}\left(a + a^\dagger\right) \quad a|\psi\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{286}|286\rangle + \sqrt{288}|288\rangle\right)$

$$a^\dagger|\psi\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{288}|288\rangle + \sqrt{289}|289\rangle\right)$$

$$\frac{\langle x\rangle}{L_0} = \frac{L_0\langle\psi|a + \hat{a}|\psi\rangle}{\sqrt{2}L_0} = \frac{2\times\sqrt{288}}{2\times\sqrt{2}} = \sqrt{144} = 12$$

Q5. A quantum mechanical particle of mass m is confined in a one-dimensional infinite potential well whose walls are located at $x = 0$ and $x = 1$. The wave function of the particle inside the well is $\psi(x) = \mathcal{N}[x\ln\ x + (1 - x)\ln\ (1 - x)]$ for some normalization constant $\mathcal{N}$. An experimentalist measures the position of the particle on an ensemble of a large number of identical systems in the same state. The mean of the outcomes is found to be $\frac{1}{n}$, where n is an integer. What is n ?

Answer: 2

Solution:

Q6. A point mass m constrained to move along the X-axis is under the influence of gravitational attraction from two-point particles each of mass M fixed at the points $(x = 0, y = a)$ and $(x = 0, y = -a)$. Find the time period of small oscillations of the mass m in units of $\pi\sqrt{\frac{a^3}{8GM}}$, where G is the universal gravitational constant.

Topic: Classical Mechanics

Subtopic: Central Force Problem

Answer: 4

Solution: $F_x = -\frac{GMm}{\left(\sqrt{a^2+x^2}\right)^2}.\cos\theta = \frac{GMm}{\left(\sqrt{a^2+x^2}\right)^2}\times\frac{x}{\sqrt{x^2+a^2}} = -\frac{GMm.x}{\left(x^2+a^2\right)^{3/2}}$ for small value of

$$m\ddot{x} = -\frac{2GMmx}{a^3\left(1+\frac{x^2}{a^2}\right)^{3/2}} = -\frac{2GMm}{a^3}x \Rightarrow \omega = \sqrt{\frac{2GM}{a^3}} \Rightarrow T = 2\pi\sqrt{\frac{4a^3}{2.4GM}} = 4\pi\sqrt{\frac{a^3}{8GM}}$$

Q7. The radial part of the electronic ground state wave function of the Hydrogen atom is

$$R_{10}(r) = \frac{1}{\sqrt{\pi a_0^3}}\exp\left(-\frac{r}{a_0}\right),$$

where a_0 is the Bohr radius. If $\langle r\rangle$ and r_{mp} denote the expectation value and the maximum probable value of the radial coordinate, respectively, compute $\frac{8}{3}\frac{\langle r\rangle}{r_{mp}}$.

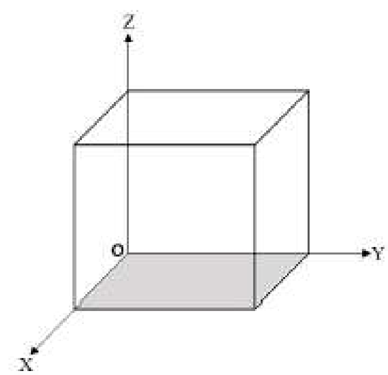
Topic: Quantum Mechanics

Subtopic: Hydrogen Atom

Answer: 4

Solution: $r_p = a_0, \langle r\rangle = \frac{3}{2}a_0 \quad \frac{8}{3}.\frac{3}{2} = 4$

Q8. A magnetic vector potential is given as $\vec{A} = 6\hat{\imath} + yz^2\hat{\jmath} + (3y+z)\hat{k}$. Find the corresponding outgoing magnetic flux through the five faces (excluding the shaded one) of a unit cube with one corner at the origin, as shown in the figure.



Topic: Mathematical Physics

Subtopic: Vectors

Answer: 0

Solution: $\phi_B = \oint \bar{B}\cdot da = \int \bar{\nabla}\times\bar{A}\cdot da$

$$\bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6 & yz^2 & 3y+z \end{vmatrix} = (3-2yz)\hat{\imath}\,, \ \ \phi_B = 0$$

Q9. In an intrinsic semiconductor at 300 K, the number density of electrons is $n_e = 2.5 \times 10^{20}\ \mathrm{m^{-3}}$. If the mobility of electrons is $\mu_e = 0.4\ \mathrm{m^2\ V^{-1}\ s^{-1}}$ and the mobility of holes is $\mu_h = 0.2\ \mathrm{m^2\ V^{-1}\ s^{-1}}$, find the conductivity in units of mho/m. Charge of a proton $e = 1.6 \times 10^{-19}$ Coulomb.

Topic: Condensed Matter Physics

Subtopic: Semiconductor

Answer: 24

Solution: $\sigma = (\eta_e \mu_e + \mu_h \eta_h)e.$

$= 2.5 \times 10^{20} \times 0.6 \times 1.6 \times 10^{-19} = 24.$

Infringe semiconductor

$\Rightarrow n_e = n_h$

Q10. An object of height 10 mm is located 150 mm to the left of a thin lens of focal length +50 mm. A second thin lens of focal length -100 mm is to be placed to the right of the first lens such that the real image of the object is located 100 mm to the right of the second lens. What should be the separation in mm between the two lenses?

Topic: Optics

Subtopic: Ray Optics

Answer: 25

Solution: $\frac{1}{v_1} - \frac{1}{u_1} = \frac{1}{f_1}$

$$\Rightarrow \frac{1}{v_1} + \frac{1}{150} = \frac{1}{50} \Rightarrow \frac{1}{v_1} = \frac{2}{150} \Rightarrow v_1 = 75\ \mathrm{mm}$$

$u_1 = -150\ \mathrm{mm}$ (due to convention).

$v_2 = 150\ \mathrm{mm}$

Separation b/w two lenses $= V_2 - V_1 = 25\ \mathrm{mm}$