

Pravegaa Education

CSIR NET-JRF · GATE · IIT-JAM · JEST · TIFR Physics

JEST 2026 — Physics

Complete, Step-by-Step Solutions

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Part A: Q1–10 (1 mark) | Part B: Q11–30 (3 marks) | Part C: Q31–40 Numerical (3 marks)

Answer Key at a Glance

Option numbers follow the official JEST 2026 response sheet (Physics, 12 April 2026, Shift 2)

Part A		Part B				Part C	
Q1	(1)	Q11	(1)	Q21	(1)	Q31	100
Q2	(2)	Q12	(2)	Q22	(3)	Q32	216
Q3	(1)	Q13	(1)	Q23	(4)	Q33	32
Q4	(2)	Q14	(4)	Q24	(3)	Q34	2
Q5	(4)	Q15	(3)	Q25	(4)	Q35	75
Q6	(2)	Q16	(1)	Q26	(2)	Q36	48
Q7	(4)	Q17	(4)	Q27	(4)	Q37	33
Q8	(1)	Q18	(4)	Q28	(1)	Q38	12
Q9	(3)	Q19	(2)	Q29	(3)	Q39	20
Q10	(4)	Q20	(1)	Q30	(4)	Q40	1.5–1.6

Part A

1-Mark MCQ (Q1–Q10) +1 / – 0.33

Q1. Group Velocity in a Dispersive Medium

Electrodynamics | Wave propagation, dispersion

An electromagnetic wave of angular frequency ω is propagating in a dispersive medium with refractive index $n \equiv n(\omega)$. If the speed of light in vacuum is c , the group velocity of the electromagnetic wave in the dispersive medium is

$$(1) \frac{c}{n + \omega \frac{dn}{d\omega}}$$

$$(2) \frac{cn}{1 + \frac{\omega}{n} \frac{dn}{d\omega}}$$

$$(3) \frac{c\omega \frac{dn}{d\omega}}{1 + \frac{\omega}{n} \frac{dn}{d\omega}}$$

$$(4) \frac{c}{1 + \frac{n}{\omega} \frac{d\omega}{dn}}$$

Solution

$$k = \frac{n(\omega)\omega}{c} \Rightarrow \frac{dk}{d\omega} = \frac{1}{c} \left(n + \omega \frac{dn}{d\omega} \right), \text{ so}$$

$$v_g = \frac{d\omega}{dk} = \frac{c}{n + \omega \frac{dn}{d\omega}}.$$

Answer:

$$(1) \quad v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$$

Q2. Critical Constants of a van der Waals Gas

Thermodynamics | Real gases

The critical volume, pressure and temperature of a van der Waals gas are V_c , P_c , and T_c , respectively. By some means the strength of the attraction between the molecules of the gas is changed. The change in V_c , P_c and T_c are

$$(1) \Delta V_c = 0, \Delta P_c = 0 \text{ and } \Delta T_c = 0$$

$$(2) \Delta V_c = 0, \Delta P_c \neq 0 \text{ and } \Delta T_c \neq 0$$

$$(3) \Delta V_c \neq 0, \Delta P_c \neq 0 \text{ and } \Delta T_c \neq 0$$

$$(4) \Delta V_c \neq 0, \Delta P_c = 0 \text{ and } \Delta T_c = 0$$

Solution

$$\text{For } \left(P + \frac{a}{V^2} \right) (V - b) = RT:$$

$$V_c = 3b, \quad P_c = \frac{a}{27b^2}, \quad T_c = \frac{8a}{27Rb}.$$

Attraction strength is the parameter a . V_c does not depend on a ; P_c and T_c are both $\propto a$.

Answer:

$$(2) \quad \Delta V_c = 0, \Delta P_c \neq 0, \Delta T_c \neq 0$$

Q3. Powers of Pauli Matrices

Mathematical Physics | *Pauli matrices*

For two Pauli matrices σ_1, σ_2 , what is the value of the expression

$$\text{Determinant}(\sigma_1^{7641}) - \text{Trace}(\sigma_2^{7641}) ?$$

The Pauli matrices are given as:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(1) -1

(2) -11

(3) 11

(4) 1

Solution

$\sigma_i^2 = I$, so for odd power $\sigma_i^{7641} = \sigma_i$.

$\det \sigma_1 = -1$, $\text{Tr} \sigma_2 = 0$. Result: $-1 - 0 = -1$.

Answer: (1) -1

Q4. Detector Efficiency

Mathematical Physics | *Binomial distribution*

A detector has a 80% particle detection efficiency. If, in a given time, 10 particles pass through the detector, what is the probability that exactly 9 particles are detected?

(1) $(0.8)^9$

(2) $2 \times (0.8)^9$

(3) $0.2 \times (0.8)^9$

(4) $(0.8)^{10}$

Solution

$$P(9) = \binom{10}{9} (0.8)^9 (0.2)^1 = 10 \times 0.2 \times (0.8)^9 = 2 \times (0.8)^9.$$

Answer: (2) $2 \times (0.8)^9$

Q5. Pressure from an Energy Scaling Law

Thermodynamics | Pressure as $-\partial U/\partial V$

The internal energy of a gas in a cubical container of length L is proportional to L^{-2} . Which power law does the pressure follow?

- (1) L^{-2} (2) L^{-3}
(3) L^{-4} (4) L^{-5}

Solution

$V = L^3$, so $U \propto V^{-2/3}$ (as for particle-in-a-box levels at fixed entropy). Then

$$P = - \left(\frac{\partial U}{\partial V} \right)_S = \frac{2}{3} \frac{U}{V} \propto \frac{L^{-2}}{L^3} = L^{-5}.$$

Answer: (4) $P \propto L^{-5}$

Q6. Ellipsoid Seen as a Sphere

Special Relativity | Length contraction

The equation of a closed surface in the lab frame is

$$16x^2 + 25y^2 + 25z^2 = 25.$$

An observer moving along the x -axis perceives the closed surface as a sphere of unit radius. If c is the speed of light in vacuum, what is the speed of the observer in the lab frame?

- (1) $\frac{4}{5}c$ (2) $\frac{3}{5}c$
(3) $\frac{9}{25}c$ (4) $\frac{16}{25}c$

Solution

$\frac{x^2}{(5/4)^2} + y^2 + z^2 = 1$: semi-axes $5/4$ along x and 1 along y, z . Only x contracts:

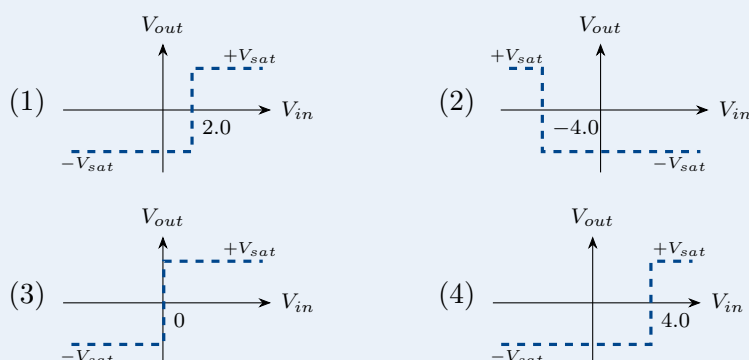
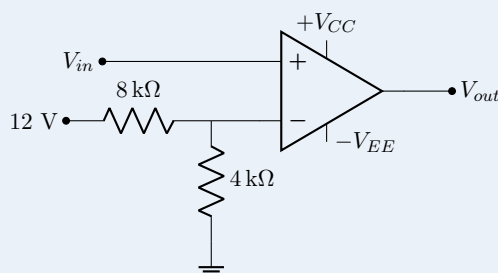
$$\frac{5}{4} \sqrt{1 - \beta^2} = 1 \Rightarrow \sqrt{1 - \beta^2} = \frac{4}{5} \Rightarrow \beta = \frac{3}{5}.$$

Answer: (2) $v = \frac{3}{5}c$

Q7. Op-Amp Comparator with a Reference Divider

Electronics | Op-amp comparator

For the given Op-Amp circuit, which of the following correctly represents the plot of V_{out} vs V_{in} ?



Solution

There is no feedback, so the op-amp is a comparator. The inverting input is held by the divider at

$$V_- = 12 \times \frac{4}{8+4} = 4 \text{ V.}$$

V_{in} drives the non-inverting input, so $V_{out} = +V_{sat}$ for $V_{in} > 4 \text{ V}$ and $-V_{sat}$ for $V_{in} < 4 \text{ V}$: a rising step at $V_{in} = 4.0 \text{ V}$.

Answer: (4) step from $-V_{sat}$ to $+V_{sat}$ at $V_{in} = 4.0 \text{ V}$

Q8. A Valid Magnetic Field

Magnetostatics | $\nabla \cdot \vec{B} = 0$

The vector $\vec{B}$ is defined as $\vec{B} = B_1 f(x)(\sin(ay)\hat{i} + \cos(ay)\hat{j}) + 2B_2 a^2 xy \hat{k}$ where a , B_1 and B_2 are constants with appropriate dimensions. For which of the following forms of the function $f(x)$ will $\vec{B}$ be a valid magnetic field?

- | | |
|----------------|---------------|
| (1) e^{ax} | (2) $\ln(ax)$ |
| (3) $\sin(ax)$ | (4) ax |

Solution

Require $\nabla \cdot \vec{B} = 0$:

$$B_1 f'(x) \sin(ay) - a B_1 f(x) \cos(ay) + \frac{\partial}{\partial z}(2B_2 a^2 xy) = B_1 \sin(ay) [f' - af] = 0.$$

So $f' = af \Rightarrow f = e^{ax}$.

Answer: (1) $f(x) = e^{ax}$

Q9. Hermiticity of a Commutator

Quantum Mechanics | *Operator algebra*

Let $\hat{A}$ and $\hat{B}$ be two operators. Consider the operator $\hat{C} = [\hat{A} + \hat{A}^\dagger, \hat{B}\hat{B}^\dagger]$, where $[\hat{x}, \hat{y}]$ represents the commutator between the operators $\hat{x}$ and $\hat{y}$. Which of the following statements about $\hat{C}$ is true?

- (1) $\hat{C}$ is Hermitian if both $\hat{A}$ and $\hat{B}$ are Hermitian.
- (2) $\hat{C}$ is always Hermitian.
- (3) $\hat{C}$ is anti-Hermitian.
- (4) $\hat{C}$ is Hermitian if $\hat{A}$ is Hermitian and $\hat{B}$ is anti-Hermitian.

Solution

$P = \hat{A} + \hat{A}^\dagger$ and $Q = \hat{B}\hat{B}^\dagger$ are Hermitian for any $\hat{A}, \hat{B}$. Then

$$[P, Q]^\dagger = (PQ - QP)^\dagger = QP - PQ = -[P, Q].$$

So $\hat{C}$ is always anti-Hermitian (it is Hermitian only in the trivial case $\hat{C} = 0$).

Answer: (3) $\hat{C}$ is anti-Hermitian

Q10. Dipole Moment about an Arbitrary Point

Electrostatics | *Dipole moment*

Two charges $+q$ and $-q$ are placed at positions $a\hat{i}$ and $a\hat{j}$. What is the dipole moment of the system with respect to a point with position vector $\vec{r}_0$?

- | | |
|--|-------------------------------------|
| (1) $q [a(\hat{i} - \hat{j}) - \vec{r}_0]$ | (2) $\sqrt{2}aq(\hat{i} - \hat{j})$ |
| (3) $q [a(\hat{i} - \hat{j}) + \vec{r}_0]$ | (4) $qa(\hat{i} - \hat{j})$ |

Solution

$\vec{p} = q(a\hat{i} - \vec{r}_0) - q(a\hat{j} - \vec{r}_0) = qa(\hat{i} - \hat{j})$. The $\vec{r}_0$ terms cancel because the net charge is zero.

Answer: (4) $qa(\hat{i} - \hat{j})$

Part B

3-Mark MCQ (Q11–Q30) +3 / -1

Q11. Far-Field Potential of a $\cos \phi$ Ring

Electrostatics | Multipole expansion

A circular ring of radius R with a line charge density $\lambda = \lambda_0 \cos \phi$ lies on the x - y plane centered at the origin. What is the approximate work done in bringing a unit positive test charge from infinity to the point (r, θ, ϕ) when $r/R \gg 1$?

- | | |
|---|---|
| (1) $\frac{\lambda_0 R^2}{4\epsilon_0 r^2} \sin \theta \cos \phi$ | (2) $\frac{\lambda_0 R^2}{4\epsilon_0 r^2}$ |
| (3) $\frac{\lambda_0 R^2}{4\epsilon_0 r^2} \cos \theta \cos \phi$ | (4) $\frac{\lambda_0 R^2}{4\epsilon_0 r^2} \cos \phi$ |

Solution

Total charge $\int_0^{2\pi} \lambda_0 \cos \phi' R d\phi' = 0$, so the dipole term leads.

$$\vec{p} = \int_0^{2\pi} \lambda_0 \cos \phi' R (\cos \phi', \sin \phi', 0) R d\phi' = \pi \lambda_0 R^2 \hat{x}.$$

$$\text{Work per unit charge} = V = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \frac{\pi \lambda_0 R^2 \sin \theta \cos \phi}{4\pi\epsilon_0 r^2} = \frac{\lambda_0 R^2}{4\epsilon_0 r^2} \sin \theta \cos \phi.$$

Answer: (1) $\frac{\lambda_0 R^2}{4\epsilon_0 r^2} \sin \theta \cos \phi$

Q12. Overlap of Unnormalised Coherent States

Quantum Mechanics | Coherent states

Consider a state $|\lambda\rangle = e^{\lambda a^\dagger}|0\rangle$ defined in a one-dimensional harmonic oscillator such that $a|\lambda\rangle = \lambda|\lambda\rangle$. Here a and $a^\dagger$ are the annihilation and creation operators respectively and $|0\rangle$ is the ground state. What is $\langle\lambda_1|\lambda_2\rangle$?

- | | |
|--|-----------------------------------|
| (1) 0 | (2) $e^{\lambda_1^* \lambda_2}$ |
| (3) $\delta(\lambda_1 - \lambda_2)$ (complex δ -function) | (4) $e^{\lambda_1^* + \lambda_2}$ |

Solution

Expand: $|\lambda\rangle = \sum_n \frac{\lambda^n (a^\dagger)^n}{n!} |0\rangle = \sum_n \frac{\lambda^n}{\sqrt{n!}} |n\rangle$. Then

$$\langle\lambda_1|\lambda_2\rangle = \sum_n \frac{(\lambda_1^* \lambda_2)^n}{n!} = e^{\lambda_1^* \lambda_2}.$$

Answer: (2) $e^{\lambda_1^* \lambda_2}$

Insight

Normalised coherent states are $e^{-|\lambda|^2/2}|\lambda\rangle$, giving $|\langle\lambda_1|\lambda_2\rangle|^2 = e^{-|\lambda_1-\lambda_2|^2}$: never orthogonal, so (1) and (3) are wrong.

Q13. $\langle 1/r \rangle$ in Hydrogen

Quantum Mechanics | *Hydrogen atom, virial theorem*

What is $\langle \frac{1}{r} \rangle$ in the state $|n, l, m\rangle$ of a hydrogen atom in terms of the Bohr radius a_0 ?

- | | |
|--------------------------|--------------------------|
| (1) $\frac{1}{n^2 a_0}$ | (2) $\frac{1}{2n^2 a_0}$ |
| (3) $\frac{1}{4n^2 a_0}$ | (4) $\frac{1}{2n a_0}$ |

Solution

Virial theorem for $V \propto 1/r$: $\langle V \rangle = 2E_n$.

$$-\frac{e^2}{4\pi\epsilon_0} \left\langle \frac{1}{r} \right\rangle = 2 \left(-\frac{e^2}{8\pi\epsilon_0 a_0 n^2} \right) \Rightarrow \left\langle \frac{1}{r} \right\rangle = \frac{1}{n^2 a_0}$$

independent of l, m .

Answer:

$$(1) \quad \frac{1}{n^2 a_0}$$

Q14. Energy Probability in an Infinite Well

Quantum Mechanics | *Stationary states, time evolution*

A particle subject to the one-dimensional infinite square well potential

$$V(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq a, \\ \infty & \text{otherwise} \end{cases}$$

is initially in the state

$$\psi(x) = \begin{cases} Ax(a-x), & \text{if } 0 \leq x \leq a, \\ 0 & \text{otherwise} \end{cases}$$

where A and a are constants. Let $P_1(t)$ be the probability that a measurement of the energy of the particle at time t yields the outcome $E_1 = \frac{\pi^2 \hbar^2}{2ma^2}$. Which of the following is true for time $t > 0$?

- (1) $P_1(t) \propto \exp \left\{ -\frac{\pi^2 \hbar^2}{2ma^2} t \right\}$
- (2) $P_1(t) \propto t^2$
- (3) $P_1(t) \propto 1 - \exp \left\{ -\frac{\pi^2 \hbar^2}{2ma^2} t \right\}$
- (4) $P_1(t)$ is independent of time

Solution

$\Psi(x, t) = \sum_n c_n \phi_n(x) e^{-iE_n t/\hbar}$, so $P_n(t) = |c_n e^{-iE_n t/\hbar}|^2 = |c_n|^2$: constant. (Here $P_1 = 960/\pi^6 \approx 0.9986$.)

Answer: (4) P_1 independent of time

Q15. Polynomial Solutions of $y'' + 2xy' + \lambda y = 0$

Mathematical Physics | *Series solution of ODEs*

Consider the solution of the differential equation $y'' + 2xy' + \lambda y = 0$ around $x = 0$. For which of the following values of λ is the solution a polynomial?

- | | |
|----------|---------|
| (1) -3 | (2) 1 |
| (3) -2 | (4) 2 |

Solution

With $y = \sum a_k x^k$:

$$a_{k+2} = -\frac{(2k + \lambda)}{(k + 2)(k + 1)} a_k.$$

The series terminates when $2k + \lambda = 0$ for some $k = 0, 1, 2, \dots$, i.e. $\lambda = 0, -2, -4, \dots$. Of the options only $\lambda = -2$ works (it gives $y = x$).

Answer: (3) $\lambda = -2$

Q16. Phase-Space Trajectory of a Relativistic Particle

Classical Mechanics | *Hamiltonian, phase space*

The Lagrangian of a particle of unit mass is given by $L = -\sqrt{1 - (dx/dt)^2} - \alpha x$, where α is a constant. The trajectory of the particle in the phase space is :

- | | |
|---------------------|----------------|
| (1) a hyperbola | (2) a circle |
| (3) a straight line | (4) an ellipse |

Solution

This is a relativistic particle ($c = 1$) in a uniform force. $p = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1 - \dot{x}^2}}$ and

$$H = p\dot{x} - L = \sqrt{1 + p^2} + \alpha x = E \Rightarrow (E - \alpha x)^2 - p^2 = 1.$$

This is a hyperbola in the (x, p) plane.

Answer: (1) a hyperbola

Q17. Angular Speed at Pericentre

Classical Mechanics | Central force motion

A satellite is orbiting in an elliptical orbit of non-zero eccentricity about a massive planet of mass M . If the angular speed of the satellite is ω_* at the point of closest distance a from the planet (where $\dot{r} = 0$ and $\ddot{r} > 0$), which of the following relations is true? G is the Gravitational constant.

$$(1) a\omega_*^2 < \frac{GM}{a^2}$$

$$(2) a\omega_*^2 = \frac{GM}{a^2}$$

$$(3) a\omega_*^2 = \frac{GM}{2a^2}$$

$$(4) a\omega_*^2 > \frac{GM}{a^2}$$

Solution

Radial equation: $\ddot{r} = r\dot{\theta}^2 - \frac{GM}{r^2}$. At pericentre $\ddot{r} > 0$, so $a\omega_*^2 > \frac{GM}{a^2}$: the satellite moves faster than circular speed and swings outward.

Answer: (4) $a\omega_*^2 > GM/a^2$

Q18. Effect of a Gauge Transformation

Electrodynamics | Gauge invariance

In an electrodynamic set-up, a gauge transformation takes the vector potential $\vec{A}$ to $\vec{A} + \vec{\nabla}f$ where $f = \alpha x^2 t^2$ with a positive constant α . Which of the following statements is correct about the electric field ($\vec{E}$) and scalar potential ϕ at $(t, x, y, z) = (-1, 1, 0, 0)$?

(1) $\vec{E}$ remains same and ϕ decreases.

(2) Both $\vec{E}$ and ϕ increase.

(3) Both $\vec{E}$ and ϕ remain same.

(4) $\vec{E}$ remains same and ϕ increases.

Solution

Accompanying change: $\phi \rightarrow \phi - \frac{\partial f}{\partial t} = \phi - 2\alpha x^2 t$. At $x = 1, t = -1$: $\Delta\phi = -2\alpha(1)(-1) = +2\alpha > 0$. Fields are gauge invariant, so $\vec{E}$ is unchanged.

Answer: (4) $\vec{E}$ same, ϕ increases

Q19. Potential Inside a Sphere with Given Boundary Values

Electrostatics | Laplace's equation, Legendre polynomials

The potential on the surface of a spherical region of unit radius in vacuum is $V(\theta) = [2 \cos^2(\theta/2) + \cos^2 \theta] V_0$, where V_0 is a constant. The electrostatic potential in the region, $r \leq 1$ unit is

- (1) $\left(\frac{2+r^2}{3} + r \cos \theta + r^2 \cos^2 \theta \right) V_0$
- (2) $\left(\frac{4-r^2}{3} + r \cos \theta + r^2 \cos^2 \theta \right) V_0$
- (3) $\left(\frac{3-r^2}{2} + r \cos \theta + r^2 \cos^2 \theta \right) V_0$
- (4) $\left(\frac{3+r^2}{4} + r \cos \theta + r^2 \cos^2 \theta \right) V_0$

Solution

$2 \cos^2(\theta/2) = 1 + \cos \theta$ and $\cos^2 \theta = \frac{1}{3} + \frac{2}{3} P_2$. So

$$V(1, \theta) = V_0 \left[\frac{4}{3} P_0 + P_1 + \frac{2}{3} P_2 \right] \Rightarrow V(r, \theta) = V_0 \left[\frac{4}{3} + r \cos \theta + \frac{2}{3} r^2 P_2 \right].$$

With $\frac{2}{3} r^2 P_2 = r^2 \cos^2 \theta - \frac{r^2}{3}$:

$$V = \left(\frac{4-r^2}{3} + r \cos \theta + r^2 \cos^2 \theta \right) V_0.$$

Check at $r = 1$: $1 + \cos \theta + \cos^2 \theta$. ✓

Answer: (2) $\left(\frac{4-r^2}{3} + r \cos \theta + r^2 \cos^2 \theta \right) V_0$

Q20. Bound Charge of a Polarised Sphere

Electrostatics in Matter | Bound charges

Consider a polarized dielectric sphere with the polarization given as $\vec{P}(r) = P_0 r \hat{i}$, where r is the distance from the centre of the sphere and P_0 is a constant. The bound charge density at a point (r, θ, ϕ) inside the sphere is

- (1) $-P_0 \sin \theta \cos \phi$
- (2) $P_0 \sin \theta \cos \phi$
- (3) $P_0 \cos \theta \cos \phi$
- (4) $-P_0 \cos \theta \sin \phi$

Solution

$$\rho_b = -\nabla \cdot \vec{P} = -P_0 \frac{\partial r}{\partial x} = -P_0 \frac{x}{r} = -P_0 \sin \theta \cos \phi.$$

Answer: (1) $-P_0 \sin \theta \cos \phi$

Q21. Hard Rods in One Dimension

Statistical Mechanics | Ergodicity, elastic collisions

Consider a system of a large even number of hard identical particles confined within a length L in one dimension which is bounded by two reflecting hard boundaries. At time $t = 0$, half of these particles have velocity $+v$ and the other half has velocity $-v$ distributed randomly among them. Assuming that all collisions are elastic, which one of the following statements is correct?

- (1) The velocity distribution of the system remains unchanged in all future times.
- (2) The velocity distribution of the system becomes the Maxwell-distribution within the time $t < L/v$.
- (3) The velocity distribution of the system becomes the Maxwell-distribution at the time $t \sim L/v$.
- (4) The velocity distribution changes later from the initial distribution but it never becomes the Maxwell-distribution.

Solution

A 1-D elastic collision between equal masses simply exchanges the two velocities; a wall reflection flips $v \rightarrow -v$. Neither creates a new speed: every particle always has speed v , and the numbers moving right and left stay balanced. The system is integrable (no thermalisation), so the distribution $\frac{1}{2}[\delta(u - v) + \delta(u + v)]$ persists forever.

Answer: (1) distribution unchanged

Q22. Free Energy of Classical Magnetic Moments

Statistical Mechanics | Paramagnetism, Langevin

There are N non-interacting magnetic moments of unit strength, fixed at certain positions in free space. These moments are subjected to a constant magnetic field $\vec{B} = B\hat{k}$ and the temperature of the system is T . If k_B is the Boltzmann constant, the Helmholtz free energy F of the system is

- (1) $Nk_B T \left[\ln \left(2 \sinh \frac{B}{k_B T} \right) - \ln \left(\frac{B}{k_B T} \right) \right]$
- (2) $-Nk_B T \left[\ln \left(2 \sinh \frac{B}{k_B T} \right) + \ln \left(\frac{B}{k_B T} \right) \right]$
- (3) $-Nk_B T \left[\ln \left(2 \sinh \frac{B}{k_B T} \right) - \ln \left(\frac{B}{k_B T} \right) \right]$
- (4) $Nk_B T \left[\ln \left(2 \sinh \frac{B}{k_B T} \right) + \ln \left(\frac{B}{k_B T} \right) \right]$

Solution

Energy $-B \cos \theta$ (unit moment); let $x = B/k_B T$. The classical single-moment partition function over orientations (azimuth integrated out) is

$$Z_1 = \int_{-1}^1 e^{x \cos \theta} d(\cos \theta) = \frac{2 \sinh x}{x}.$$

$F = -Nk_B T \ln Z_1 = -Nk_B T [\ln(2 \sinh x) - \ln x]$. (Keeping the 2π from ϕ only adds a constant $-Nk_B T \ln 2\pi$.)

Answer: (3) $-Nk_B T [\ln(2 \sinh x) - \ln x]$

Insight

$M = -\partial F/\partial B = N \left(\coth x - \frac{1}{x} \right)$: the Langevin function, as expected.

Q23. Net Charge Density with Moving Electrons

Special Relativity | Transformation of charge density

An infinite material has electrons moving freely in the background of fixed positive charges. Initially the average charge density of electrons (ρ^-) and that of the positive charges (ρ^+), satisfy $\rho^+ = -\rho^- = \rho_0$. What is the average net charge density of the material if an electric field $\vec{E}$ is turned on resulting in a steady velocity $\vec{u}$ of the electrons throughout the material? c is the speed of light.

- (1) 0
- (2) $\rho_0 \left(1 - \frac{1}{(1 - u^2/c^2)^{3/2}} \right)$
- (3) $\rho_0 \left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right)$
- (4) $\rho_0 \left(1 - \frac{1}{\sqrt{1 - u^2/c^2}} \right)$

Solution

The electron gas, in its own rest frame, keeps its equilibrium density ρ_0 (proper density). In the lab frame, where the electrons move with speed u , their spacing along $\vec{u}$ is Lorentz contracted, so

$$\rho_{\text{lab}}^- = -\gamma\rho_0, \quad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}.$$

The fixed ions are unaffected: $\rho^+ = \rho_0$. Hence

$$\rho_{\text{net}} = \rho_0 - \gamma\rho_0 = \rho_0 \left(1 - \frac{1}{\sqrt{1 - u^2/c^2}} \right),$$

a small net negative charge density.

Answer: (4) $\rho_0 \left(1 - 1/\sqrt{1 - u^2/c^2} \right)$

Insight

This is the same Lorentz contraction that makes a current-carrying wire appear charged to a moving observer (Purcell's derivation of magnetism).

Q24. A Fourier-Sine Integral

Mathematical Physics | Contour integration

What is the value of the integral $\int_0^\infty \frac{k \sin(kr)}{k^2 + m^2} dk$ for $r > 0$?

(1) $\frac{1}{4\pi r} e^{-mr}$

(2) $\frac{\pi r}{1 + m^2 r^2}$

(3) $\frac{\pi}{2} e^{-mr}$

(4) $\frac{1}{r\sqrt{1 + m^2 r^2}}$

Solution

The integrand is even: $I = \frac{1}{2} \text{Im} \int_{-\infty}^\infty \frac{k e^{ikr}}{k^2 + m^2} dk$. Close in the upper half-plane (Jordan's lemma, $r > 0$); pole at $k = im$ with residue $\frac{im e^{-mr}}{2im} = \frac{e^{-mr}}{2}$:

$$\int_{-\infty}^\infty = 2\pi i \cdot \frac{e^{-mr}}{2} = i\pi e^{-mr} \Rightarrow I = \frac{\pi}{2} e^{-mr}.$$

Answer:

(3) $\frac{\pi}{2} e^{-mr}$

Insight

This is the integral behind the Yukawa potential e^{-mr}/r as the Fourier transform of $1/(k^2 + m^2)$.

Q25. An Integro-Differential Equation

Mathematical Physics | Linear ODEs

What is the general solution of the equation $\frac{df(t)}{dt} = f(t) + 6 \int f(t) dt$?

(1) $f(t) = Ae^{2t} + Be^{-3t}$

(2) $f(t) = Ae^{-2t} + Be^{-3t}$

(3) $f(t) = Ae^{2t} + Be^{3t}$

(4) $f(t) = Ae^{-2t} + Be^{3t}$

Solution

Differentiate: $f'' = f' + 6f$. Characteristic equation $s^2 - s - 6 = (s - 3)(s + 2) = 0 \Rightarrow s = 3, -2$.

$$f(t) = Ae^{-2t} + Be^{3t}.$$

Answer:

(4) $Ae^{-2t} + Be^{3t}$

Q26. Commuting with L_x^2 **Quantum Mechanics** | Angular momentum algebra

An operator A is expressed in terms of angular momentum operators as:

$$A = L^2 + aL_z^2 + b(L_+L_- + L_-L_+),$$

where $L_{\pm} = L_x \pm iL_y$ and a, b are arbitrary constants. Under what condition does A commute with L_x^2 ?

(1) $a = b$

(2) $a = 2b$

(3) $2a = b$

(4) $a = -b$

Solution

$L_+L_- + L_-L_+ = 2(L_x^2 + L_y^2) = 2(L^2 - L_z^2)$. Therefore

$$A = (1 + 2b)L^2 + (a - 2b)L_z^2.$$

L^2 commutes with L_x^2 , but $[L_z^2, L_x^2] \neq 0$. So we need $a - 2b = 0$.

Answer: (2) $a = 2b$

Q27. Turning a Bright Fringe into a Dark One**Optics** | Young's double slit

In Young's double slit set-up, the screen is placed at a distance $D = 150$ cm from the slits. At a point P on the screen, the second maximum is being observed. What is the minimum amount by which D needs to be changed such that the point P is now at a minimum?

(1) 80 cm

(2) 20 cm

(3) 50 cm

(4) 30 cm

Solution

At fixed P (height y), path difference $\Delta = yd/D \propto 1/D$. Initially $\Delta = 2\lambda$.

- Increase D to reach 1.5λ : $D' = 150 \times \frac{2}{1.5} = 200$ cm (change 50 cm).
- Decrease D to reach 2.5λ : $D' = 150 \times \frac{2}{2.5} = 120$ cm (change 30 cm).

Minimum change = 30 cm.

Answer: (4) 30 cm

Q28. Displaced Oscillator in Momentum Representation

Quantum Mechanics | *Momentum-space representation*

Which of the following Hamiltonians correctly represents a one-dimensional simple harmonic oscillator of mass m , frequency ω and minimum of the potential at $x = a$?

- (1) $\mathcal{H} = -\frac{\hbar^2}{2}m\omega^2 \frac{\partial^2}{\partial p^2} + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 a^2 - i\hbar m\omega^2 a \frac{\partial}{\partial p}$
- (2) $\mathcal{H} = -\frac{\hbar^2}{2}m\omega^2 \frac{\partial^2}{\partial p^2} + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 a^2$
- (3) $\mathcal{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial p^2} + \frac{1}{2}m\omega^2 (p - a)^2$
- (4) $\mathcal{H} = -\frac{\hbar^2}{2}m\omega^2 \frac{\partial^2}{\partial p^2} + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 a^2 + i\hbar m\omega^2 a \frac{\partial}{\partial p}$

Solution

In p -space $\hat{x} = i\hbar \partial/\partial p$ and $\hat{x}^2 = -\hbar^2 \partial^2/\partial p^2$. Expand

$$\frac{1}{2}m\omega^2(x - a)^2 = \frac{1}{2}m\omega^2 x^2 - m\omega^2 a x + \frac{1}{2}m\omega^2 a^2.$$

Substituting:

$$\mathcal{H} = \frac{p^2}{2m} - \frac{\hbar^2 m\omega^2}{2} \frac{\partial^2}{\partial p^2} - i\hbar m\omega^2 a \frac{\partial}{\partial p} + \frac{1}{2}m\omega^2 a^2.$$

Answer: (1) with $-i\hbar m\omega^2 a \partial_p$

Q29. Entropy of Two Bosons in an Oscillator

Statistical Mechanics | *Microcanonical counting*

Two identical bosons in a one-dimensional harmonic oscillator potential (with angular frequency ω) have total energy $E = 100\hbar\omega$. What is the entropy of this state if k_B is the Boltzmann constant?

- | | |
|--------------------|--------------------|
| (1) $k_B \ln(198)$ | (2) $k_B \ln(100)$ |
| (3) $k_B \ln(50)$ | (4) $k_B \ln(99)$ |

Solution

$$E = (n_1 + \frac{1}{2})\hbar\omega + (n_2 + \frac{1}{2})\hbar\omega = 100\hbar\omega \Rightarrow n_1 + n_2 = 99.$$

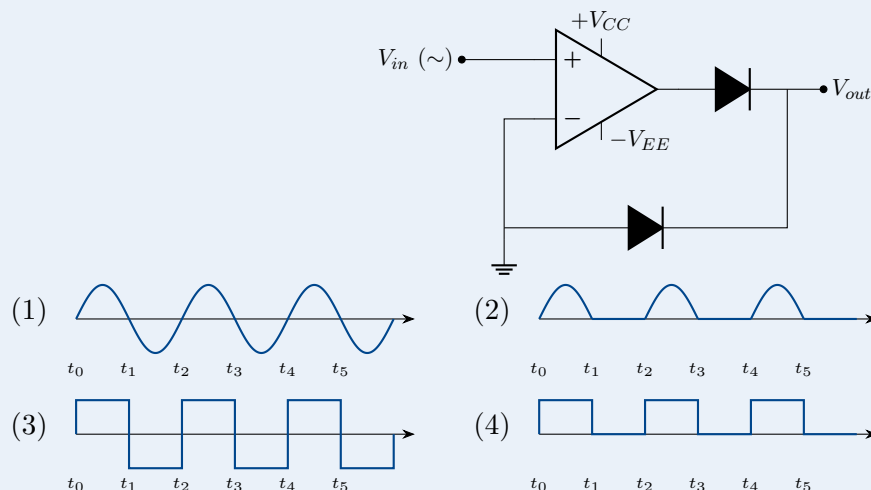
For identical bosons only unordered pairs count: $n_1 \leq n_2$ gives $n_1 = 0, 1, \dots, 49$ (99 is odd, so no $n_1 = n_2$). $\Omega = 50$, so $S = k_B \ln 50$.

Answer: (3) $k_B \ln 50$

Q30. Comparator with Diode Output

Electronics | Op-amp comparator, diodes

Consider the Op Amp circuit with a sinusoidal voltage input. What is the correct output waveform of this circuit?



Solution

With the inverting input grounded and no feedback, the op-amp is a zero-crossing detector: output $+V_{sat}$ when $V_{in} > 0$ (t_0-t_1 , t_2-t_3 , ...) and $-V_{sat}$ when $V_{in} < 0$.

- Positive half-cycles: the series diode conducts, so $V_{out} \approx +V_{sat}$ (flat top, not sinusoidal).
- Negative half-cycles: the series diode is reverse biased; the lower diode ties V_{out} to ground level, so $V_{out} \approx 0$.

Output: unipolar rectangular pulses between t_0-t_1 , t_2-t_3 , t_4-t_5 .

Answer: (4) positive square pulses, zero otherwise

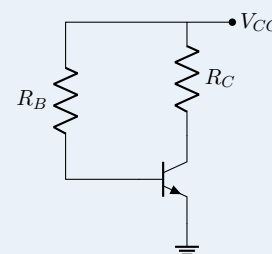
Part C

Numerical Answer (Q31–Q40) +3/0

Q31. Current Gain with $V_{CE} = V_{BE}$

Electronics | BJT biasing

Consider the given circuit with $V_{CC} = 10V$, $R_B = 1M\Omega$, and $R_C = 10k\Omega$. It is given that $V_{CE} = V_{BE}$, where V_{CE} is the collector-emitter voltage and V_{BE} is the base-emitter junction voltage. What is the gain β of the transistor?



Solution

Both resistors are fed from V_{CC} :

$$I_B R_B = V_{CC} - V_{BE}, \quad I_C R_C = V_{CC} - V_{CE}.$$

With $V_{CE} = V_{BE}$ the right-hand sides are equal, so $I_C R_C = I_B R_B$ and

$$\beta = \frac{I_C}{I_B} = \frac{R_B}{R_C} = \frac{10^6}{10^4} = 100.$$

Answer: $\boxed{\beta = 100}$

Q32. Re-alignment of Three Planets

Classical Mechanics | *Kepler's third law*

Three planets are orbiting a star S_* in coplanar circular orbits of radii R , $4R$ and $9R$. The time period of the planet closest to the star is 1 year. At some instant the star and the planets are collinear. Ignoring the motion of the star, after how many years will the same aligned configuration of the planets occur again?

Solution

$T \propto R^{3/2}$: $T_1 = 1$, $T_2 = 8$, $T_3 = 27$ years. The *same* configuration recurs when every planet has completed a whole number of orbits:

$$t = \text{LCM}(1, 8, 27) = 216 \text{ years.}$$

Even allowing a common overall rotation, the relative angles $(\omega_1 - \omega_3)t$ and $(\omega_2 - \omega_3)t$ must be multiples of 2π : $\frac{26}{27}t \in \mathbb{Z}$ and $\frac{19}{216}t \in \mathbb{Z}$, whose smallest common solution is again $t = 216$.

Answer: $\boxed{216 \text{ years}}$

Q33. Measurement Changes the Odds

Quantum Mechanics | *Sequential measurements*

In a two-dimensional Hilbert space, two Hermitian non-commuting operators A and B have non-degenerate normalized eigenstates, which satisfy $A|\psi_A^i\rangle = a_i|\psi_A^i\rangle$ and $B|\psi_B^i\rangle = b_i|\psi_B^i\rangle$ where $i = 1, 2$. The relations:

$$|\psi_A^1\rangle = \frac{1}{\sqrt{5}}|\psi_B^1\rangle + \frac{2}{\sqrt{5}}|\psi_B^2\rangle; \quad |\psi_B^1\rangle = \frac{1}{\sqrt{5}}|\psi_A^1\rangle - \frac{2}{\sqrt{5}}|\psi_A^2\rangle$$

are given. After a measurement of B is done on the state $|\psi_A^2\rangle$, what is the percentage probability of finding the system in the state $|\psi_A^1\rangle$?

Solution

From the given relations, $|\langle\psi_B^1|\psi_A^2\rangle|^2 = \frac{4}{5}$ and $|\langle\psi_B^2|\psi_A^2\rangle|^2 = \frac{1}{5}$; also $|\langle\psi_A^1|\psi_B^1\rangle|^2 = \frac{1}{5}$, $|\langle\psi_A^1|\psi_B^2\rangle|^2 = \frac{4}{5}$.

$$P = \underbrace{\frac{4}{5}}_{B \rightarrow b_1} \cdot \frac{1}{5} + \underbrace{\frac{1}{5}}_{B \rightarrow b_2} \cdot \frac{4}{5} = \frac{8}{25} = 32\%.$$

Without the intermediate B measurement the answer would be 0: the measurement destroys orthogonality.

Answer: $\boxed{32\%}$

Q34. Relativistic Motion under Constant Force**Special Relativity** | *Hyperbolic motion*

A particle of mass m_0 is initially at rest in an inertial frame and is subjected to a constant force m_0g along the y -axis. Its motion is governed by the equation:

$$\frac{dp}{dt} = m_0g,$$

where p is the relativistic momentum. If $y(t)$ depicts the distance covered by the particle in time t as $y\left(t = \frac{\sqrt{8}c}{g}\right) = n\frac{c^2}{g}$, where c is the speed of light, what is n ?

Solution

$p = m_0gt = \gamma m_0v \Rightarrow v = \frac{gt}{\sqrt{1 + (gt/c)^2}}$. Integrating,

$$y(t) = \frac{c^2}{g} \left[\sqrt{1 + \left(\frac{gt}{c}\right)^2} - 1 \right].$$

At $gt/c = \sqrt{8}$: $y = \frac{c^2}{g}(\sqrt{9} - 1) = 2\frac{c^2}{g}$.

Answer: $n = 2$

Q35. Spin Flip under $\vec{S}_1 \cdot \vec{S}_2$ **Quantum Mechanics** | *Addition of angular momenta, time evolution*

A system of two spin-1/2 particles is in the state $|\uparrow\downarrow\rangle$ for time $t < 0$. At $t = 0$, an interaction Hamiltonian $\hat{H} = C\hat{\mathbf{S}}^{(1)} \cdot \hat{\mathbf{S}}^{(2)}$ is turned on, where C is a constant of the appropriate dimension and $\hat{\mathbf{S}}^{(1)}$ and $\hat{\mathbf{S}}^{(2)}$ are the spin angular momentum operators of the first and second particle respectively. What is the probability in percentage to find the system in the state $|\downarrow\uparrow\rangle$ at time $t = \frac{2\pi}{3C\hbar}$?

Solution

$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,0\rangle)$, with $E_T = \frac{1}{4}C\hbar^2$ and $E_S = -\frac{3}{4}C\hbar^2$ (so $\Delta E = C\hbar^2$).

$$\langle\downarrow\uparrow|\psi(t)\rangle = \frac{1}{2} \left(e^{-iE_T t/\hbar} - e^{-iE_S t/\hbar} \right) \Rightarrow P = \sin^2\left(\frac{\Delta E t}{2\hbar}\right) = \sin^2\left(\frac{C\hbar t}{2}\right).$$

At $t = \frac{2\pi}{3C\hbar}$: $P = \sin^2(\pi/3) = \frac{3}{4} = 75\%$.

Answer: 75%

Q36. Action on the Extremal Path**Classical Mechanics** | *Calculus of variations*

Consider the action $S = \int_0^2 [(dx/dt)^2 + 6x] dt$. If the boundary conditions are such that $x(0) = 0$, and $x(2) = 6$, what is the value of the action evaluated on the extremal path?

Solution

Euler-Lagrange: $\frac{d}{dt}(2\dot{x}) = 6 \Rightarrow \ddot{x} = 3 \Rightarrow x = \frac{3}{2}t^2 + c_1t$. $x(2) = 6 + 2c_1 = 6 \Rightarrow c_1 = 0$.

$$S = \int_0^2 (9t^2 + 9t^2) dt = 18 \cdot \frac{8}{3} = 48.$$

Answer: $S = 48$

Q37. Positive Predictive Value of a Test**Mathematical Physics** | *Bayes' theorem*

A diagnostic test for a particular disease detects it with an efficiency of 99.99% in infected people. It also detects a false positive result in 0.02% of people who are not infected. If one person in 10000 in the general population is infected, what is the probability (in nearest integer percent) that someone who tests positive is actually infected?

Solution

$$P(I|+) = \frac{(0.9999)(10^{-4})}{(0.9999)(10^{-4}) + (0.0002)(0.9999)} = \frac{10^{-4}}{10^{-4} + 2 \times 10^{-4}} = \frac{1}{3} \approx 33\%.$$

Answer: 33%

Q38. Fixing a Lagrangian Parameter**Classical Mechanics** | *Euler-Lagrange equation*

The Lagrangian of a particle in one dimension is $L = 10\dot{x}^2 + 6\dot{x}t^2 + \alpha xt$ where $\dot{x} \equiv dx/dt$ and α is a constant. The particle starts moving at time $t = 0$ from the position $x(0) = 0.5$ with $\dot{x}(0) = 0.5$ in appropriate units. For what value of α , is the position of the particle at time t given by $x(t) = \frac{t+1}{2}$?

Solution

$\frac{d}{dt}(20\dot{x} + 6t^2) = \alpha t \Rightarrow 20\ddot{x} + 12t = \alpha t$. For $x = \frac{t+1}{2}$, $\ddot{x} = 0$, so $\alpha = 12$ (the initial conditions are automatically satisfied).

Answer: $\alpha = 12$

Q39. Leakage between Concentric Spheres**Electrodynamics** | *Steady currents, resistance*

Two concentric conducting spheres of radii 4 m and 5 m respectively, enclose a medium of conductivity σ (in units of $m^{-1}\Omega^{-1}$) between their surfaces. The potential difference between the surfaces of the spheres is maintained at V (in volts). In this setup, there will be a current I (in amperes) between the spheres. What is the characteristic length scale $I/(4\pi\sigma V)$ of the system in metres?

Solution

$$R = \int_a^b \frac{dr}{\sigma 4\pi r^2} = \frac{1}{4\pi\sigma} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{4\pi\sigma} \cdot \frac{1}{20}.$$

$$I = \frac{V}{R} = 4\pi\sigma V \cdot 20 \Rightarrow \frac{I}{4\pi\sigma V} = \frac{ab}{b-a} = 20 \text{ m}.$$

Answer: 20 m**Q40. Newton's Rings in a Liquid****Optics** | *Thin-film interference*

Consider a Newton's rings experiment with a biconvex lens of diameter 20 cm and maximum thickness $60 \mu\text{m}$ placed on a flat reflecting surface, illuminated by normal incident light of wavelength 600 nm. When we change the medium of the experiment from air to a liquid medium, the fringe separation reduces by 20%. What the refractive index of the liquid medium?

Solution

Ring radii $r_m = \sqrt{m\lambda R/\mu} \propto \mu^{-1/2}$, so all fringe separations scale as $\mu^{-1/2}$ (the lens data fix R , which cancels in the ratio).

$$\frac{1}{\sqrt{\mu}} = 0.8 \Rightarrow \mu = \frac{1}{0.64} = 1.5625.$$

Answer: $\mu \approx 1.56$ (accepted range 1.5–1.6)